RUSSELL'S PARADOX

VOLKER PECKHAUS
Institut für Philosophie / Universität Erlangen-Nürnberg
Bismarckstr. 1, D-91054 Erlangen, Germany
VRPECKHA@PHIL.UNI-ERLANGEN.DE

Alejandro R. Garciadiego. *Bertrand Russell and the Origins of the Set-theoretic 'Paradoxes'.* Basel, Boston and Berlin: Birkhäuser Verlag, 1992. Pp. xxix, 264. US$77.50. Spanish edition: *Bertrand Russell y los orígenes de las "paradojas" de la theoría de conjuntos.* Madrid: Alianza Editorial, 1992. Pp. 237.

The year 1903 marks a seminal step in the development of formal logic and the foundations of mathematics. The subequent transformations of logic were ironically caused by a catastrophical result published by Bertrand Russell in his book *The Principles of Mathematics*, whose tenth chapter is devoted to "The Contradiction". The most famous specimen of this contradiction is the paradox which today bears Russell's name: Consider the set of all sets that do not contain themselves as an element. If this set does contain itself it contains at least one set that does contain itself, against the supposition. If it does not contain itself, it does not contain all sets that do not contain itself, again against the supposition. The same year Gottlob Frege published the second volume of his *Grundgesetze der Arithmetik*,[1] confessing that Russell's paradox could be formulated in his logical system; thus the most elaborated system of mathematical logic of that time was endangered.

It was only after the publication of the paradoxes that the new symbolic logic, connected with names like George Boole, William Stanley Jevons, Charles S. Peirce, Ernst Schröder, Gottlob Frege, and Giuseppe Peano, gained interest in wider circles of philosophers and mathematicians. The paradoxes initiated the vivid development of proof theory in the beginning of the twentieth century, they gave important impulses for modern axiomatics, and they were still present in the background of the foundational crisis in mathematics in the 1920s.

[1] Gottlob Frege, *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet*, Vol. 2 (Jena: Hermann Pohle, 1903; reprinted: Hildesheim: Olms, 1962).

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Alejandro Garciadiego's book is devoted to clarifying the prehistory and the context of these publications. At the same time it illustrates the conceptual emergence of a new type of contradiction which stands for an inconsistency arising from sound propositions by accepted methods of reasoning. This type of contradiction is now called "paradox" in the English-speaking world, although the term traditionally refers to apparent contradictions due to fallacies in reasoning or to inappropriate assumptions. Already in 1907 the Göttingen mathematician Ernst Zermelo suggested that the much more precise term "antinomy" should be used for the new type of contradictions; it is now standard in German terminology. In a postcard to the Göttingen philosopher Leonard Nelson, he wrote that the word "paradox" refers to a proposition which clashes with the common opinion, but in which nothing of an inner contradiction can be found.[2] Garciadiego prefers to use "neutral terms" (p. xi n.1), and he has good reasons, since one of the major topics of his book is the emergence of the conviction that the problematic sets and inconsistencies arising in Cantor's set theory represent the new type of contradiction. Furthermore, Bertrand Russell, who adhered to neo-Hegelian positions in his early years of philosophical authorship, was working on antinomies in the Kantian sense, which differed from logical antinomies. In Kant antinomies refer to the dialectical antithesis of two propositions, which both seem to be well founded by dogmatic reasoning. In the critical use of reason such antinomies can be resolved. Hence, they are only apparent contradictions.

Garciadiego's book is based on his PH.D. thesis defended at the Institute for the History and Philosophy of Science and Technology of the University of Toronto in 1983. For the present version it was substantially revised not least to change the scope of the intended audience towards "all students interested in the history and philosophy of ideas, not only to those who specialize in mathematics" (p. xi). For professional historians, philosophers and mathematicians he provides a new interpretation of the origins of the set-theoretic paradoxes, especially the role played in this story by Russell. His target is a revision of the so-called standard interpretation of the origins of the set-theoretic paradoxes as it is sketched in Ivor Grattan-Guinness's Prologue to Garciadiego's book: "Cantor found the one [paradox] concerning the greatest cardinal in the 1890s, and soon afterwards Burali-Forti discovered the corresponding for the ordinals. Then around 1900 Russell showed that

[2] Zermelo's postcard to Nelson, dated 22 December 1907, Nelson Papers, Archiv der sozialen Demokratie, Bonn-Bad Godesberg, quoted in Volker Peckhaus, *Hilbertprogramm und Kritische Philosophie. Das Göttinger Modell interdisziplinärer Zusammenarbeit zwischen Mathematik und Philosophie* (Göttingen: Vandenhoeck & Ruprecht, 1990), p. 104.

the set of all sets which do not belong to themselves led to a paradox" (p. ix).

Gardiadiego has divided his exposition into five chapters. Chapter I is devoted to the "Antecedents" of Russell's invention. He gives the mathematical essentials of Cantor's transfinite arithmetic, especially his theory of cardinal and ordinal numbers.

In Chapter II Garciadiego criticizes the standard interpretation of the origins of the set-theoretical paradoxes. He shows that the first "paradoxes" arising from set-theory, Cesare Burali-Forti's paradox of the greatest ordinal and Cantor's paradox of the greatest cardinal, were simply *reductio ad absurdum* arguments in order to prove that some concepts are not valid in set theory. In 1897 Burali-Forti tried to prove that it is impossible to order the order types in general and the ordinal numbers in particular (p. 24). Cantor tried to show that the sets of all cardinals and of all ordinals have to be removed from set theory with the help of his distinction between consistent and inconsistent multiplicities, as communicated to Dedekind in 1899 (p. 35).

In Chapter III the philosophical and mathematical background of Russell's *Principles of Mathematics* is given. Garciadiego discusses Russell's early biography, the years of his intellectual formation, his acceptance of neo-Hegelianism (F. H. Bradley, B. Bosanquet, J. McT. E. McTaggart) during his university studies at Cambridge, his early studies on the foundations of geometry and of arithmetic in a neo-Hegelian spirit, his step-by-step dissociation from these positions beginning in 1897 due to his reading of Leibniz, Dedekind and his first acquaintance with Cantor's set theory. Garciadiego hints also at the influences of Alfred North Whitehead's *Universal Algebra*,[3] which is said to have had no effect on Russell. According to Garciadiego, Russell's early efforts to found the basic concepts of arithmetic upon those of the logical calculus[4]—which can be found in an unsuccessful attempt of 1898 to write a book on the principles of mathematics—are due to Whitehead's influence.

In Chapter IV Garciadiego gives a plausible historical reconstruction of the actual way in which Russell discovered the paradoxes. By carefully analyzing the preserved drafts of the *Principles* and related material from the Bertrand Russell Archives at Hamilton, Ont., Garciadiego again rejects some of the myths which can be found in the literature, e.g., that it was above all Peano's influence which forced Russell towards the final version of the book. "Unfortunately," Garciadiego writes, "Russell's own emphasis of the influence of Peano on his thinking has hidden the tremendous influence of Cantor"

[3] *A Treatise of Universal Algebra with Applications*, Vol. I (Cambridge: at the U. P., 1898).

[4] For Garciadiego "this seems to be the foundation of Russell's logicist program" (p. 68), although Russell did not develop this programme in the subsequent years.

(p. 82). By analyzing the manuscript sources of Russell's various attempts over approximately six years to write a book on the principles of mathematics Garciadiego shows that Russell produced the final manuscript of the *Principles* in the period after meeting Peano, i.e. from November 1900 until January 1903. He wrote it in three stages, rewriting especially Part I, which contains the chapter on "The Contradiction" during the final stage between May 1902 and January 1903. This can be regarded as an indication that Russell finally had become convinced of the extraordinary character of his paradox, and that this conviction was due to Frege's reaction after having been informed by Russell. Garciadiego illustrates the different stages of writing the *Principles* by printing the outlines of tables of contents from the Russell papers.

In Chapter V Garciadiego discusses the relation of the so-called non-logical or semantical paradoxes to the logical ones discovered by Bertrand Russell. He especially treats Berry's paradox, the König–Zermelo paradox, Richard's paradox, and finally the polemics within the London Mathematical Society concerning Zermelo's well-ordering theorem. His result is, contrary to the standard interpretation, that "in general, ... these non-Mathematical inconsistencies did not originate directly from the 'logical' ones" (p. 153). Garciadiego does not deal with the effect of indirect influences, which, however, I take to be quite important. Only after the beat of the drum of the double publication of the set-theoretic paradoxes, paradoxes (and not only fallacies) became a topic of discussion in broader circles, and minds became open for the new type of contradictions. Of course, his result is not valid for all semantical paradoxes. As I have shown elsewhere,[5] Grelling's paradox was a direct offspring of the discussion of Russell's paradox of non-predicability in the circle of Leonard Nelson. Garciadiego does not treat Grelling's paradox in detail since it was not mentioned by Russell himself. The paradox of non-predicability, i.e., the paradox which arises from the notion of predicates which cannot be predicated of themselves, is only mentioned in a quotation (p. 105) from a folio of Russell's written in 1901, presumably the first formulation of this paradox to which Russell later gave prominence in his attempt to give a general formulation of "The Contradiction" (*PoM*, p. 102). Nevertheless, the quoted passage is clearly *not* "the earliest statement of Russell's contradiction of the *class of all classes which are not members of themselves*" as Garciadiego claims (p. 105). It would have been worthwhile if Garciadiego had given some hints on the relation between the two paradoxes.

[5] "The Genesis of Grelling's Paradox", in *Logik und Mathematik. Frege-Kolloquium 1993*, ed. Ingolf Max and Werner Stelzner (Berlin and New York: Walter de Gruyter, 1995), pp. 269–80.

In the appendix Garciadiego edits parts of Russell's correspondences with Alys Russell, G. E. Moore, D. Hilbert, C. Burali-Forti, G. G. Berry, A. N. Whitehead, G. H. Hardy, and E. H. Moore, unfortunately without giving any references. The book is closed by an extensive bibliography (containing several misprints) and a valuable index.

In conclusion, Garciadiego's book gives a vivid introspection into the emergence of philosophical concepts of eminent importance for logic and the foundations of mathematics. Today it can be read as a handy companion to Volume 3 of *The Collected Papers of Bertrand Russell* containing papers "Toward the *Principles of Mathematics*".[6]

[6] This review was originally published in Spanish in *Mathesis*, 11 (1995): 285–90.

[1] All page numbers are to the second edition, even though what is at issue is the theory as presented in the first edition. Russell added a new introduction to the second edition, but left the main work largely unchanged.

Throughout the rest of this review, page numbers not otherwise identified will be from *Russell et le cercle des paradoxes*.