

THE PARADOXES AND THE THEORY OF TYPES

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Philippe de Rouilhan. *Russell et le cercle des paradoxes*. Paris: Presses Universitaires de France, 1996. Pp. 319. 198 FF.

Russell introduced his theory of types in *Principia Mathematica* as a method of resolving certain paradoxes, and as one that had a “consonance with common sense which makes it inherently credible” (*PM* I: 37).¹ Russell saw the theory of types given in *Principia* as an alternative to Set Theory, a basis for mathematics which was purely logical, which did not yield contradictions, and which was not *ad hoc*. Philippe de Rouilhan’s book, *Russell et le cercle des paradoxes*, is an account of Russell’s confrontation with the paradoxes, and of his proposed solution to them. The book is extremely thorough and contains one of the most detailed discussions to date of Russell’s substitutional theory of classes and relations and the theory of types

advocated in *Principia Mathematica*. De Rouilhan has an impressive mastery not only of the mathematical intricacies of the theory, but of the philosophical issues driving Russell. For those scholars who are interested in the development of Russell's mathematical philosophy, his book is extremely important.

De Rouilhan begins by reviewing the various paradoxes which had troubled Russell after his *Principles of Mathematics*, including his famous paradox of 1901. He sees these paradoxes as "serious" paradoxes, for they resist analysis, their resolutions themselves seem paradoxical, and they require us to give up certain intuitions or things we have thought of as data or even fundamental principles: the solutions of the paradoxes cause us to renounce certain fundamental principles (p. 10).

De Rouilhan thinks, as others have, that Russell's solution to the paradoxes caused him to renounce some of his earlier principles. He sees Russell's theory of denotation and his acceptance of the vicious-circle principle as major changes in Russell's thinking from *Principles*. However, he also detects a continuity in Russell's work on logic and the unrestricted variable. While he holds that Russell did change his position on several matters during the development of the ramified type theory of *Principia*, he does not believe that the changes are as great as other commentators have said they are. The second part of this book, particularly the last chapter on the theory of types, is de Rouilhan's most important contribution to the literature on Russell's logic.

Some parts of this work suffer because the author has not examined the manuscripts available in the Russell Archives, most of which are now published in the *Collected Papers*. This lacuna is especially noticeable in the sections on the theory of denoting and the 1905–06 substitutional theory. With respect to the former, de Rouilhan largely ignores the argument in "On Fundamentals", despite the fact that he mentions it in a footnote. He has also not looked at the "FN" manuscript.² Had he examined these early papers, de Rouilhan would have seen that the arguments given in "On Denoting" have an even closer relevance to the 1901 paradox than he gives them.

With respect to the substitutional theory, the problem is a little more serious. De Rouilhan does an excellent job with what he has (the papers which were published in *Essays in Analysis*), but he is left to speculate, with Hylton

² Published as "Fundamental Notions", *Papers* 4.

(1980),³ as to why Russell abandoned the substitutional theory. In this regard, he would have done well to look at some of the unpublished manuscripts in the Russell Archives and the work of Gregory Landini.⁴ Russell's manuscript "The Paradox of the Liar", which is crucial to the development of this theory and the move to the theory of types, is not even mentioned. I would surmise that the section on the substitutional theory was written several years before the book was published.

Russell's theory of types is notoriously difficult to understand, and there have been several attempts⁵ to reconstruct the system so that it is both faithful to Russell's stated philosophy of logic (as given in the Introduction to *Principia Mathematica*) and yet philosophically consistent and logically rigorous. Most commentators find it necessary to modify Russell's theory somewhat, and to be selective about which philosophical remarks in the Introduction they will emphasize. Since Gödel's important paper of 1944,⁶ it has been recognized that there is a tension between Russell's usual realism and the constructivism which appears to be behind the vicious-circle principle. One formulation of the vicious-circle principle is that "whatever contains an apparent variable must not be a possible value of that variable;"⁷ another that "whatever involves all of a collection must not be one of the collection" (*PM* I: 37). If this is taken to apply to sets, and the principle taken to apply to the definition of the set, then it looks as though the set is thought of as something created out of its definition, implying some kind of mental construction. (And indeed, some logicians, such as Brouwer, in fact took it this way.) But this sort of constructivism, as Gödel pointed out, seems alien to Russell's realism. Some recent commentators, most noticeably Warren Goldfarb,⁸ have defended the use of the vicious-circle principle in conjunction with a realism concerning propositional functions by arguing that the propositional functions are themselves intensional entities and thus can contain

³ Peter Hylton, "Russell's Substitutional Theory", *Synthese*, 45 (1980): 1–31.

⁴ Gregory Landini, "New Evidence Concerning Russell's Substitutional Theory of Classes", *Russell*, n.s. 9 (1989): 26–42.

⁵ See, for example, Kurt Schütte, *Beweistheorie* (Berlin: Springer, 1960); Alonzo Church, *Introduction to Mathematical Logic* (Princeton: Princeton U. P., 1956); Charles Chihara, *Ontology and the Vicious Circle Principle* (Ithaca: Cornell U. P., 1973); and Alonzo Church, "A Comparison of Russell's Resolutions of the Semantical Antinomies with That of Tarski", *Journal of Symbolic Logic*, 41 (1976): 747–60.

⁶ Kurt Gödel, "Russell's Mathematical Logic", in Schilpp.

⁷ This formulation occurs in "Mathematical Logic as Based on the Theory of Types", *American Journal of Mathematics*, 1908; reprinted in *LK*, p. 75.

⁸ Warren Goldfarb, "Russell's Reasons for Ramification", in C. Wade Savage and C. Anthony Anderson, eds., *Rereading Russell: Essays in Bertrand Russell's Metaphysics and Epistemology* (Minneapolis: U. of Minnesota P., 1989).

quantifiers. Goldfarb's interpretation involves thinking of the theory of types as being a theory of objects of different logical types.

Philippe de Rouilhan is somewhat more cautious, and, I believe, more faithful to Russell's intentions in his interpretation of Russell's philosophy of mathematics. De Rouilhan sees Russell as trying to maintain his universalism concerning logic while embracing the vicious-circle principle as the key to the correct solution of the paradoxes. Russell's instinct was for a universal logic. This was, as many have pointed out, the key to Russell's account of logic,⁹ and some, including Goldfarb, have thought that Russell wittingly or unwittingly was forced to give up the doctrine of the unrestricted variable because of the theory of types.

Russell attempted to maintain the universality with the substitutional theory, which treated expressions for classes, relations and functions as incomplete symbols. These symbols were not taken to stand for entities and thus the type restriction on substitution of the symbols is justified without the supposition of a corresponding type differentiation among entities. The theory of types as it was first formulated with the substitutional theory was a theory of the symbolism, not a theory of the symbolized, and thus the individual variable was the only general and completely unrestricted variable. All other variables were to be explained away in terms of constructions from the individual variables.

De Rouilhan thinks that Russell came to abandon the unrestricted variable, but suggests that he was abandoning the universality only provisionally, with the understanding that a full theory would vindicate universality. In a very helpful section, de Rouilhan distinguishes three types of constructivism (pp. 216–17). What he calls “constructivism₁” is the methodology of “logical constructivism” where classes are eliminated in favour of other entities (and can in some sense be thought to be “constructed” from them); “constructivism₂” is the constructivism which de Rouilhan argues is involved in Russell's notion of the construction of higher propositions from lower ones. It is the view that the initial building-blocks of the world are the individuals and the concepts and relations which apply to them. The entities of higher types exist only in so far as they are composed of these building-blocks, constructed from these in the sense that they contain no further entities. Elementary propositions are constructed from individuals by the repeated operations of predication, negation, implication and substitution, the first-order ones constructed from these by the operations of generalization on individual variables

⁹ See, for example, Nino Cocchiarella, “The Development of the Theory of Logical Types and the Notion of a Logical Subject in Russell's Early Philosophy”, and Nicholas Griffin, “Russell on the Nature of Logic (1903–1913)”, in *Synthese*, 45 (1980): 70–115, 117–88.

and the other previously mentioned operations, those of the second order constructed from these with the addition of the operation of generalization on function variables. De Rouilhan contrasts this sort of constructivism with what he calls “constructivism₃”, which holds that the higher order objects are mental constructs and do not have an existence independent of human minds. De Rouilhan points out that it is this last kind of constructivism which Gödel attacked and which appears to be at odds with Russell's realism.

De Rouilhan's discussion of these constructivisms occurs in his discussion of the theory presented in “Mathematical Logic as Based on the Theory of Types”, a theory which he calls an “intermediate theory”. It is true that in this work Russell does appear to countenance not just different types of symbols, but different ontological types, individuals and then propositions of the various orders. De Rouilhan goes a long way towards reconciling Russell's commitment to the unrestricted variable and his adoption of the vicious-circle principle for restriction on proper substitutional instances of quantified variables. Russell himself advanced this theory tentatively, and replaced it with the theory outlined in *Principia Mathematica*. Here there are no longer propositions of differing ontological types, rather “propositions” (here understood to be expressions) are incomplete symbols—there are no longer held to be ontological counterparts to the expressions (they are constructions, in de Rouilhan's sense).

When it comes to reconstructing the logical system of *Principia*, though, Rouilhan does see the theory of types as being a theory of more than the symbols. That is, he sees Russell as being committed to a hierarchy of objects of different ontological types, individuals at the base, and propositions and propositional functions as constructions₂. He recognizes that he has modified Russell's system somewhat in developing his presentation of it. There are two points where his reconstruction differs from Gregory Landini's,¹⁰ and which have a bearing on the question of what sort of ontology Russell was committed to in *Principia*. First, de Rouilhan takes the circumflexed variables in the way Church has: to be understood as λ -abstraction, taking this to be a variablebinding operator. He also extends (and must if he is to do this) the bound variables (those Russell calls “apparent variables”) to include functions which are not predicative. Thus his statement of the axiom of reducibility (p. 256) is

¹⁰ Gregory Landini, “Reconciling *PM*'s Ramified Type Theory with the Doctrine of the Unrestricted Variable of the *Principles*”, in A. D. Irvine and G. A. Wedeking, eds., *Russell and Analytic Philosophy* (Toronto: U. Toronto P., 1993), and “Will the Real *Principia* Please Stand Up? Reflections on the Formal Logic of the *Principia*”, in Ray Monk and Anthony Palmer, eds., *Bertrand Russell and the Origins of Analytical Philosophy* (Bristol: Thoemmes P., 1996).

$$(\forall f)(\exists g)(\forall x_1 \dots x_k)(fx_1 \dots x_k \equiv g!x_1 \dots x_k)$$

Gregory Landini has criticized both these moves as at variance with the logic of *Principia*. He argues that Russell did not regard the circumflex as a variable-binding operator, and in particular would not have countenanced such functions as

$$\lambda z (\forall x)(fx, z)$$

Landini has also argued that Russell was right to reject quantification over non-predicative functions, and that the Axiom of Reducibility should best be seen as a schema of the following sort

$$(\exists g) (\forall x_1 \dots x_k)(g!x_1 \dots x_k \equiv A)$$

rather than as an open formula with real variables which only becomes a statement when they are properly bound (as de Rouilhan and Church and others have seen it). Landini holds that once we understand that propositional functions and propositions are constructions, (in de Rouilhan's sense), then we can see that the theory of types is really a theory of the symbols and not also an ontological theory of the symbolized. Though he includes functions of different orders as entities in his intended interpretation of the type theory of *Principia*, de Rouilhan also appears to recognize in places that Russell had as a goal their elimination:

Si les propositions et fonctions ne figurent pas à l'inventaire du monde des *Principia*, c'est dans l'idée qu'une analyse plus profonde (fondée sur la méthode de substitution et la théorie du jugement) aurait permis de les faire apparaître pour ce qu'elles sont: des symboles incomplets, des fictions logiques, rien: «Une "proposition" [...] n'est pas du tout *une* entité» [PM I: 44]; «une fonction propositionnelle n'est rien» [PLA, in LK, p. 230]. (P. 254)

If propositions and functions don't figure in the inventory of the world of *Principia*, it is in the idea that a deeper analysis (based on the method of substitution and the theory of judgement) would allow them to be seen for what they are: incomplete symbols, logical fictions, nothing: "A proposition ... is not a single entity at all" (PM I: 44); "A propositional function is nothing" (PLA, in LK, p. 230; *Papers* 8: 20–2)