

Russell and Zeno's arrow paradox

by Paul Hager

ON RUSSELL'S ACCOUNTS of Zeno's Arrow Paradox, Gregory Vlastos comments that there "seem to be almost as many Zenos in Russell as there are Russells."¹ Zeno of Elea is, in fact, a philosopher whom Russell often discusses,² and Vlastos' remark appears to be amply justified when we note that, for example, in his 1903 *Principles of Mathematics* Russell was asserting that Weierstrass had vindicated Zeno and established that "we live in an unchanging world and ... the arrow, at every moment of its flight, is truly at rest",³ yet by his 1914 *Our Knowledge of the External World* Russell had adopted the opposite view of the arrow that at "a given instant, it is where it is ... but we cannot say that it is at rest at the instant."⁴ Reversals such as this are, of course, the basis

¹ G. Vlastos, "A Note on Zeno's Arrow", in R.E. Allen and D.J. Furley, eds., *Studies in Presocratic Philosophy* (New York: Humanities Press, 1975), II: 184-200 (at 199).

² The principal sources are "Recent Work in the Philosophy of Mathematics", *The International Monthly*, 4 (July 1901), reprinted under the title "Mathematics and the Metaphysicians" in *Mysticism and Logic* (London: Longmans, Green, 1918); *The Principles of Mathematics* (Cambridge, 1903); "The Philosophy of Bergson", *The Monist*, 22 (July 1912), republished with a reply by H. Wildon Carr and a rejoinder from Russell as *The Philosophy of Bergson* (London, Glasgow and Cambridge: Bowes and Bowes, 1914); and *Our Knowledge of the External World* (London and Chicago: Open Court, 1914), Lectures v and vi. The Bergson lecture of 1912 was later included in the chapter on Bergson in Russell's *A History of Western Philosophy* (New York: Simon and Schuster, 1945; London: Allen and Unwin, 1946). This chapter was severely cut in the British second edition of 1961, the material on Zeno being part of the omissions.

³ P. 347. See also "Mathematics and the Metaphysicians", p. 63, for the same claim.

⁴ P. 142. (Page references to *Our Knowledge* are to the rev. 1926 Allen and Unwin ed.)

of the famous C.D. Broad remark that "Mr. Russell produces a different system of philosophy every few years...."⁵

Vlastos himself highlights Russell's apparent changes of mind about Zeno's arrow by suggesting that different Russellian accounts of the paradox ascribe different assumptions to Zeno. Thus Vlastos views the *Our Knowledge* account as imputing to Zeno the central assumption "that there are consecutive instants", yet much later, in the *History of Western Philosophy*, Russell had, according to Vlastos, produced another interpretation which centres on the different Zenonian assumption "that there can be no motion unless there are instantaneous states of motion."⁶

Since none of Zeno's writings have survived, our knowledge of the paradoxes of motion derives from secondary sources. This scantiness of direct evidence has led to a proliferation of interpretations and reconstructions of the arguments, so Russell would perhaps not be alone if he had, indeed, changed his interpretation of the Arrow Paradox as frequently as Vlastos suggests. Nonetheless, despite the evidence of vacillation outlined above, I will argue that Russell consistently maintained a single interpretation of the Arrow Paradox. The apparent differences and changes noted above will be seen to be differences of *emphasis* stemming from developments in Russell's doctrines concerning space and time, developments which can, in fact, be shown to underlie all of the major changes in Russell's philosophy.⁷

Accordingly, I will present my reconstruction of what Russell took Zeno's argument to be and then show how the differences emphasized by Vlastos are more apparent than real. Of course, Russell himself never set out the complete argument explicitly. However, the subsequent discussion will show the textual fidelity of my reconstruction.

RECONSTRUCTION OF RUSSELL'S ACCOUNT OF ZENO'S ARROW PARADOX

- (1) Finite intervals of spaces and times consist of series of points and instants.
- (2) The series of points and instants are either finite or infinite.

⁵ In J.H. Muirhead, ed., *Contemporary British Philosophy*, First Series (London: Allen and Unwin, 1924), p. 79.

⁶ Vlastos, p. 199. Vlastos is apparently unaware that the *History of Western Philosophy* account is taken directly from "The Philosophy of Bergson", which was published in 1912. Likewise he admits to being unable to date "Mathematics and the Metaphysicians" beyond its appearance in J. Newman, ed., *The World of Mathematics* (New York: Simon and Schuster, 1956). (See Vlastos, p. 200.)

⁷ The evidence for this general view of Russell's philosophy is given in my doctoral dissertation, "Continuity and Change in the Development of Russell's Philosophy" (Department of Traditional and Modern Philosophy, University of Sydney, 1987).

- (3) The series of points and instants can't be infinite (since such a view leads to contradictions).

Therefore:

- (4) Finite intervals of spaces and times consist of finite series of points and instants. [(1), (2), (3).]
- (5) But successive (discontinuous) occupation of finite series of points and instants is not sufficient to constitute the essential continuity of motion through such intervals.

Therefore:

- (6) The essential continuity of motion entails that a moving object must have throughout its motion (and hence at every instant and point) something to supply the continuity which an object at rest lacks (call this something a "state of motion"). [(4), (5).]
- (7) At each instant the arrow in flight simply is where it is (Zeno's Platitude).
- (8) An arrow that is where it is at an instant does not move during the instant (otherwise the instant would have parts).
- (9) But an arrow that does not move during an instant has no state of motion at the instant.

Therefore:

- (10) The arrow in flight has at each instant no state of motion. [(7), (8), (9).]

Therefore:

- (11) The arrow has no motion. [(6), (10).]

The *core argument* attributed to Zeno by Russell has (7), (8) and (9) as premisses entailing (10) as conclusion. I call this the "core argument" because not only does Russell think it valid, but also he holds each of the premisses to be true, i.e. he holds (10) to be true. Where Russell dissents from Zeno is, of course, in respect of (11), but to derive (11) we require the further assumption (6) and, therefore, (4) and (5). (4) and (6) are the two key assumptions that Russell expressly attributes to Zeno as the basis of his support for (11)—they are also the two assumptions that Russell is most concerned to deny. We can call the argument (4)–(10) to the conclusion (11), Zeno's "expanded argument". (1)–(3) are given as reasons why Zeno (and others) might adopt the key assumption (4). I distinguish between the *core argument* (which Russell accepts) and the *expanded argument* (which he rejects) because the appearance of vacillation on his part is reinforced by him sometimes supporting Zeno, yet later dissenting from him.

Some of these premisses and/or conclusions in Russell's interpretation of Zeno require further comment. In the course of this discussion footnotes will locate the crucial premisses of the core and expanded arguments in Russell's writings, thereby establishing the accuracy of

the above reconstruction.

(1) and (2) I take to be, respectively, plausible and obvious.⁸

(3) is a premiss that Russell himself had strongly supported in his early Kantian-Hegelian idealist phase. It stems from the notorious difficulties with the notion of infinity that have so strongly influenced the course of Western philosophy, e.g. the tradition stretching from Aristotle to Leibniz and Kant that denies actual infinity while allowing potential infinity.⁹ During his excursion into idealism Russell had spent a lot of effort consigning space, time, motion, matter and change to the realm of appearance, as distinct from reality, on the basis of deriving contradictions about them from considerations that depended largely on traditional views about infinity.¹⁰ Little wonder then that, convinced by the work of Weierstrass and Cantor that a theory of infinity that evades the paradoxes and difficulties of the philosophers is viable, Russell subsequently became almost missionary in his espousal of the denial of (3). Some commentators would claim that Zeno's first two paradoxes of motion support (3), i.e. they attempt to show that the series of points and instants constituting finite intervals of space and time can't be infinite.¹¹ This is not, however, an interpretation that Russell supports.¹²

(4) is the premiss that Russell emphasizes in *Our Knowledge of the External World*, but it also appears in his other discussion of the Arrow Paradox, often under the guise of consecutive points and instants.¹³ If the series of points or instants is finite, then there is a *next* point or instant with the intervals between successive points or instants being infinitesimals. But if, as Russell believed modern mathematics implied, the series of points or instants is infinite and compact, then, of course, there is *no* next point or instant and no need to postulate infinitesimals. Hence Russell's rejection of (4).

(5) and (6) are premisses that have impressed many philosophers con-

vinced that *continuity* is the most essential characteristic of motion. For example, Bergson denied that points and instants had anything to do with an account of motion. Leibniz held that moving bodies have an internal force or activity to constitute their state of motion, and hence continuity. The self-evident incompatibility of a discontinuous finite series of points and instants with the perceived continuity of motion is the basis of Bergson and Leibniz's views. Russell discusses this point in terms of the problem of how change of position can occur in the infinitesimal intervals between instants.¹⁴ His answer, of course, is that the supposed difficulty is due merely to a failure of the *imagination* to satisfactorily comprehend the nature of compact series.¹⁵

(7), which Russell dubs "Zeno's Platitude" and takes to be the Eleatic's central insight, is echoed in Russell's characterization of motion in the light of modern mathematics: "Motion consists merely in the fact that bodies are sometimes in the one place and sometimes in another, and that they are at intermediate places at intermediate times."¹⁶

(8) is, perhaps, behind Russell's rather startling claim that "Weierstrass, by strictly banishing from mathematics the use of infinitesimals, has at last shown that we live in an unchanging world."¹⁷ However, I take this to be a provocative gloss on the perfectly reasonable point that the arrow's non-moving at any given *instant* is completely irrelevant to its moving or resting during some *interval* containing the given instant.

(9) I take to be obvious, and (10) is, of course, what Russell takes Zeno to have definitely established.¹⁸ (10) is also central to the mathematical theory of motion that Russell wishes to support against the mainstream philosophical account. As against Zeno, however, the mathematical theory denies the implausible (11).

Having established that the above interpretation of Zeno's Arrow Paradox was the one consistently maintained by Russell in all of his writings on the subject, we need to return to the apparent vacillations

⁸ See, e.g., *Our Knowledge*, pp. 142 and 183.

⁹ Russell frequently discusses the traditional difficulties with the notion of infinity, e.g. "Mathematics and the Metaphysicians", pp. 66ff.; *Principles*, Ch. XLIII; *Our Knowledge*, Lecture VI.

¹⁰ For a good sample of this see *My Philosophical Development*, Ch. 4.

¹¹ See, e.g., G.E.L. Owen, "Zeno and the Mathematicians", reprinted in Allen and Furlley, II: 143-65.

¹² *Our Knowledge*, pp. 173-9.

¹³ In "Mathematics and the Metaphysicians" (p. 65), (4) appears as the assumption that there are *consecutive* instants separated by an infinitesimal interval. In the *Principles* it appears as the assumption that there is an infinitesimal difference between *successive* values of a continuous variable such as time (pp. 351-3). For (4) in *Our Knowledge*, see pp. 135, 174, 179, 183, etc. In "The Philosophy of Bergson" it appears as the assumption of a *next* instant (p. 18).

¹⁴ See, e.g., *Our Knowledge*, p. 179.

¹⁵ *Our Knowledge*, Lecture V. (5) and (6) are set out in these principal sources: "Mathematics and the Metaphysicians", p. 65; *Principles*, Ch. XLII, esp. p. 352; *Our Knowledge*, pp. 136, 144-5, 157ff.; "The Philosophy of Bergson", pp. 17-18.

¹⁶ "Mathematics and the Metaphysicians", p. 66. See also *Our Knowledge*, p. 144. (7) appears in these principal sources: "Mathematics and the Metaphysicians", pp. 65-6; *Principles*, p. 351; *Our Knowledge*, pp. 142 and 179; "The Philosophy of Bergson", pp. 17-19.

¹⁷ *Principles*, p. 347. See also "Mathematics and the Metaphysicians", p. 63. (8) appears in these principal sources: "Mathematics and the Metaphysicians", p. 65-6 (implicit); *Principles*, p. 351; *Our Knowledge*, p. 179; "The Philosophy of Bergson", pp. 18-19.

¹⁸ "Mathematics and the Metaphysicians", p. 63; *Principles*, p. 351; *Our Knowledge*, pp. 136, 142, 179; "Philosophy of Bergson", p. 19.

and differences between Russell's various accounts, which have been highlighted, as we saw, by Vlastos. For a start, if our version of the argument *was* consistently maintained by Russell, why wasn't this clear to Vlastos? Vlastos, in fact, considers four passages where Russell discusses the Arrow Paradox, two of them involving detailed discussion and the other two a more cursory treatment. In the former case, i.e. in *The Principles of Mathematics* and *Our Knowledge of the External World*, Russell never sets out the full argument in one place. Rather in both instances he presents more than the *core argument* (7)–(10) but less than the *expanded argument* (4)–(11) in the one central location (both referenced by Vlastos), with the rest of the argument spread over several chapters or sections, but identifiable on careful reading by the references to Zeno and the problems posed by his Arrow Paradox.

In the latter case, since Russell wasn't attempting a detailed account of the Arrow Paradox in either instance, we can't expect to find the full argument. Nevertheless in the *History of Western Philosophy* account, reprinted from *The Philosophy of Bergson*, the *expanded argument*, (4)–(11), is clearly set out in detail,¹⁹ while (1)–(3) are implicit in the succeeding discussion of Bergson's denial that objects in motion occupy any points.²⁰ In "Mathematics and the Metaphysicians", originally published in an American magazine in 1901 under the title "Recent Work in the Philosophy of Mathematics", Russell was responding to editorial insistence that the article should be "as romantic as possible".²¹ We can hardly expect a scholarly consideration of the Arrow Paradox in such a piece. In the event, the refutation of the two key assumptions (4) and (6) is stressed,²² while other parts of the argument are fairly explicit in discussion of infinity, continuity and infinitesimals.

What then of the apparent vacillations by Russell highlighted by Vlastos? Firstly there is the claim that Russell on different occasions imputes different central assumptions to Zeno. As we have seen, Vlastos maintains that whereas the *Our Knowledge* account hinges on the assumption "that there are consecutive instants", the *History* one is based on the different assumption "that there can be no motion unless there are instantaneous states of motion". The first of these assumptions is equivalent to our (4)—a check on the context shows that Russell's "there are consecutive instants" is a gloss on "the view that a

finite part of time consists of a finite series of successive instants."²³ The second assumption is clearly another version of our (6). As I pointed out, (4) and (6) are the two key assumptions that Russell attributes to Zeno. They *are* key assumptions precisely because, according to Russell, modern developments in mathematics have demonstrated their falsity, thus finally answering Zeno's challenge. That Russell stresses (4) in one context and (6) in another in no way demonstrates that he has changed his account of the Arrow Paradox. Indeed (4) and (6) are close relatives since the move from (4) to (6) merely requires (5), a thesis that many philosophers have taken to be obviously true. Of course, the modern theory of continuity entails that (5) is irrelevant, since the series of points and instants are infinite rather than finite.

Secondly, and more seriously, there are the contradictory claims noted earlier: in 1903 that "the arrow, at every moment of its flight, is truly at rest"; and in 1914 that the arrow at an instant "is where it is ... but we cannot say that it is at rest at the instant." However, there is a simple explanation of this discrepancy which leaves our account of the Zeno argument unscathed. The point is merely that Russell is working with different definitions of "rest" in the two contradictory quotations. In the *Principles* he notes that "rest" "is a loose and ambiguous expression"²⁴ and distinguishes "rest throughout an interval" from "instantaneous rest".²⁵ Clearly the arrow at every moment of its flight is at rest in the second sense. In *Our Knowledge*, however, Russell can consistently deny that the arrow is "at rest at the instant" because on the same page he explicitly defines "rest" so as to rule out the second sense of the term:

... we cannot say that it is at rest at the instant, since the instant does not last for a finite time, and there is not a beginning and end of the instant with an interval between them. Rest consists in being in the same position at all the instants throughout a certain finite period, however short; it does not consist simply in a body's being where it is at a given instant.²⁶

Moreover, far from its being a matter of caprice that Russell should adopt different definitions of "rest" in 1903 and 1914, the matter is entirely consistent with changes in his views about space and time in the interim. Russell's philosophy falls into three distinct phases, each

¹⁹ "The Philosophy of Bergson", pp. 17–19 [(4)–(5) in the paragraph beginning "Bergson's position ..." and (6)–(11) in the paragraph beginning "Zeno assumes ..."] (The corresponding pages in *History of Western Philosophy* [1946], are 832–3.)

²⁰ *Ibid.*, p. 19 (*History of Western Philosophy* [1946], pp. 833–4).

²¹ *Mysticism and Logic*, 1917 preface.

²² "Mathematics and the Metaphysicians", pp. 69 and 65 respectively.

²³ *Our Knowledge*, p. 179.

²⁴ *Principles*, p. 265.

²⁵ *Ibid.*, p. 473.

²⁶ *Our Knowledge*, p. 142.

shaped by his views about space and time: 1900–13, Platonism; 1914–19, Empiricism; and 1920 onwards, Modified Empiricism. In the Platonist phase, points and instants are paradigms of Platonic entities, and particles have irreducible triangular relations to points and instants. Thus in 1903 Russell believed that, at each moment of its flight, the arrow had irreducible relations to real points and instants—no different from the instantaneous relations pertaining to an arrow at rest. Hence it is hardly surprising that the *Principles* should talk of the arrow being at instantaneous rest during its flight. All of Russell's discussions of the Arrow Paradox fall within the Platonist phase except for the account in *Our Knowledge* coming at the start of the Empiricist phase. This phase is marked by a move from irreducible relations between Platonic entities as the foundations for philosophical analysis, the replacement being relations between sense-data and sensibilia. Since points and instants are now logical constructions, the notion of the arrow being at instantaneous rest becomes a mathematical abstraction having no simple connection with the ultimate furniture of the world. Accordingly Russell no longer had the same motivation for singling out the notion of instantaneous rest, and therefore dropped it. However, none of this denies the truth of (8).

Russell has a reputation for erratically changing his mind on important issues. He is also regarded by many as a clear-minded writer who set down his ideas in a straightforward, easily understood manner. I hope my paper has cast doubt on both of these views.

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