Design of Minimum BER Linear Space-Time Block Codes for MIMO Systems Equipped with Zero-Forcing Equalizer–Correlated Channels

DESIGN OF MINIMUM BER LINEAR SPACE-TIME BLOCK CODES FOR MIMO SYSTEMS EQUIPPED WITH ZERO-FORCING EQUALIZER–CORRELATED CHANNELS

BY

LISHA WANG, B.Eng.

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AUTHOR: Lisha Wang
B.Eng., (Communications Engineering)
Beijing University of Posts and Telecommunications, Beijing, China

SUPERVISOR: Prof. Kon Max Wong

CO-SUPERVISOR:

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To Those Whom I Love And Those Who Love Me

Abstract

In this thesis, we consider a coherent MIMO system, emphasizing on the simplicity of implementation at both the code generator and the receiver. Specifically, we consider the transmission of a space-time block code (STBC) that is a linear combination of coding matrices weighted by the information symbols through a receiver-correlated flat-fading channel and received by a linear ZF detector. Our target is the design of a code which, while maintaining full data-transmission rate, minimizes the asymptotic average (over all the random channel coefficients) bit error probability of an ZF detector. To this end, we first ensure that the full data rate of symbols is maintained, and then, based on the BER for 4-QAM signals, we derive the conditions for optimal codes and establish a code structure that minimizes the asymptotic average bit error probability. We also prove that the diversity gain of our $M \times N$ MIMO system is $N - M + 1$. The resulting optimum code structure requires the individual coding matrices to be mutually orthogonal when vectorized and is related to covariance matrix of correlated channel. The first optimum structural characteristics of the coding matrices is described as trace-orthogonal. A new approach to express expected value of random correlated channel has been proposed as well. Our optimum code structure under statistically known CSI has the similar structure under fully known CSI [1]. From simulation results we can see that advantage of optimum code over uncoded

system is more apparent as channel correlation is higher.

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Acronyms

BER	Bit Error Rate
DFT	Discrete Fourier Transform
MIMO	Multi-Input Multi-Output
QAM	Quadrature Amplitude Modulation
SNR	Signal-to-Noise Ratio
STBC	Space-Time Block Code
ZF	Zero-Forcing

Glossary of Symbols

$\mathbf{a}$	Column vector $\mathbf{a}$
$\mathbf{A}$	Matrix $\mathbf{A}$
$(\cdot)^T$	The transpose of a vector or matrix
$(\cdot)^*$	The complex conjugate of a vector or matrix
$(\cdot)^H$	The Hermitian of a vector or matrix
ln	Natural logarithm
$\mathrm{E}[\cdot]$	The expectation operator
$\mathrm{Re}(\cdot)$	Real part of the variable in the bracket
$\mathrm{Im}(\cdot)$	Imaginary part of the variable in the bracket
$\mathrm{Tr}(\cdot)$	The trace operator
$\otimes$	Kronecker product
j	$\sqrt{-1}$

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Chapter 1

Introduction

1.1 Background

1.1.1 MIMO Communication Systems

MIMO is an acronym that stands for Multiple Input Multiple Output. It is an antenna technology that is used both in the transmitter and the receiver for wireless radio communication. MIMO technology has attracted attention in wireless communications because it offers significant increases in data throughput and link range without additional bandwidth or transmission power. It achieves this by higher spectral efficiency (more bits per second per hertz of bandwidth) and link reliability or diversity (reduced the effect of fading). Because of these properties, MIMO is an important part of modern wireless communication standards such as IEEE 802.11n (Wifi), 4G, 3GPP Long Term Evolution, WiMAX and HSPA+.

The basic principle of MIMO is to take advantage of multi-path. MIMO uses multiple antennas to send multiple parallel signals (from transmitter). In an urban environment, these signals will bounce off trees, buildings, etc. and continue on their way to their destination (the receiver) but in different directions and at different arrival time. Multi-path occurs when the different signals arrive at the receiver at various times. With MIMO, the receiving end uses an algorithm or special signal processing to sort out the multiple signals to produce one signal that has the originally transmitted data. By transmitting multiple data streams in different channels at the same time and collecting multipath signals with multiple sensors, MIMO delivers simultaneous speed, coverage, and reliability improvements.

Spatial Multiplexing

Spatial multiplexing (SM) is a transmission technique in MIMO wireless communication to transmit independent and separately encoded data signals, so-called streams, from each of the multiple transmitter antennas. Arogyaswami Paulraj and Thomas Kailath proposed the concept of spatial multiplexing (SM) using MIMO in 1993. Their US Patent No. 5,345,599 on Spatial Multiplexing issued 1994 [2] emphasized applications to wireless broadcast. In 1996, Greg Raleigh and Gerard J. Foschini refined new approaches to MIMO technology, considering a configuration where multiple transmitter antennas are co-located at one transmitter to improve the link throughput effectively [3]. Bell Labs was the first to demonstrate a laboratory prototype of spatial multiplexing in 1998, where spatial multiplexing is a principal technology to improve the performance of MIMO communication systems [4].

In spatial multiplexing, a high rate signal is split into multiple lower rate streams and each stream is transmitted from a different transmitter antenna in the same frequency channel. If these signals arrive at the receiver antenna array with sufficiently

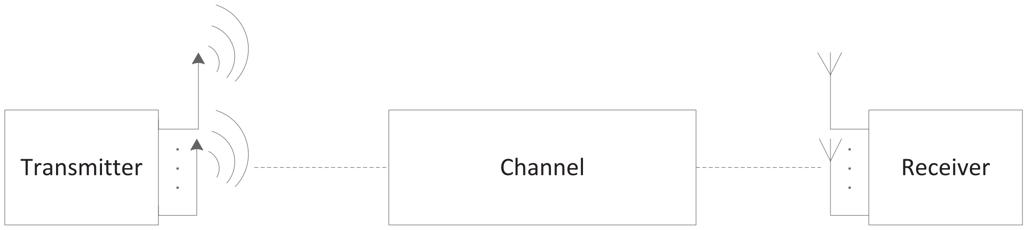


Figure 1.1: MIMO communication systems

different spatial signatures, the receiver can separate these streams into (almost) parallel channels. Spatial multiplexing is a very powerful technique for increasing channel capacity at higher signal-to-noise ratios (SNR).

A basic MIMO communication system is made up of transmitter, channel and receiver (Fig 1.1).

1.1.2 Transmitter: Space-Time Block Codes

At the transmitter, multiple data streams are emitted from the transmitter antennas with independent and appropriate weightings, which is called precoding, such that the link throughput is maximized at the receiver. Space-time block coding is one of the precoding techniques to transmit multiple copies of a data stream across a number of antennas and to exploit the various received versions of the data to improve the reliability of data transfer. The fact that the transmitted signal often traverses a potentially difficult environment with scattering, reflection, refraction and so on and may then be further corrupted by thermal noise in the receiver means that some of the received copies of the data will be 'better' than others. This redundancy results in a higher chance of being able to use one or more of the received copies to correctly decode the received signal. In fact, spacetime coding combines all the copies of the received signal in an optimal way to extract as much information from each of them as possible.

Proposed by Vahid Tarokh, Nambi Seshadri and Robert Calderbank, spacetime codes [5] (STCs) achieve significant error rate improvements over single-antenna systems. Their original scheme was based on trellis codes but the simpler block codes were utilized by Siavash Alamouti [6], and later Vahid Tarokh, Hamid Jafarkhani and Robert Calderbank [7] to develop spacetime block-codes (STBCs). STC involves the transmission of multiple redundant copies of data to compensate for fading and thermal noise in the hope that some of them may arrive at the receiver in a better state than others. In the case of STBC in particular, the data stream to be transmitted is encoded in blocks, which are distributed among spaced antennas and across time.

1.1.3 Channel: Channel State Information

In wireless communications, channel state information (CSI) refers to known channel properties of a communication link. This information describes how a signal propagates from the transmitter to the receiver and represents the combined effect of, for example, scattering, fading, and power decay with distance. The CSI makes it possible to adapt transmissions to current channel conditions, which is crucial for achieving reliable communication with high data rates in multiantenna systems.

CSI needs to be estimated at the receiver and usually quantized and fed back to the transmitter through a separate channel. Therefore, extra bandwidth is needed for the sharing of CSI between the receiver and transmitter.

There are basically two levels of CSI, namely instantaneous CSI and statistical CSI.

i **Instantaneous CSI** (or short-term CSI) means that the current channel conditions are known, which can be viewed as knowing the impulse response of a channel. This gives an opportunity to adapt the transmitted signal to the impulse response and thereby optimize the received signal for spatial multiplexing or to achieve low bit error rates.

ii **Statistical CSI** (or long-term CSI) means that a statistical characterization of the channel is known. This description may include, for example, the type of fading distribution, the average channel gain, the line-of-sight component, and statistics of the spatial correlation. As with instantaneous CSI, this information can be used for partial transmission optimization.

CSI acquisition is practically limited by how fast the channel conditions are changing. In fast fading systems where channel conditions vary rapidly under the transmission of a single information symbol, only statistical CSI is reasonable. On the other hand, in slow fading systems instantaneous CSI can be estimated with reasonable accuracy and used for transmission adaptation for some time before being outdated.

1.1.4 Receiver: Detection/Decoding

At the receiver, received signals will be recovered and detected using decoding/detection methods.

Maximum Likelihood (ML) Decoding

ML is the decoding, which compares the distance between received symbols to all the possible combination sequence. The one with the shortest distance will be selected as decoded symbols. This method is often computational prohibitive in practice.

Equalizer

The equalizer is a device that attempts to reverse the distortion suffered by a signal transmitted through a channel. Its purpose is to reduce intersymbol interference to allow the recovery of the transmitted symbols. It may be a simple linear filter or a complex algorithm. Linear Equalizer processes the incoming signal with a linear filter. Compared to ML decoding, the following two linear equalizers are much simpler for implementation:

i **Minimum Mean-Square Error(MMSE) Equalizer**: designs the filter to minimize $E[|e|^2]$, where e is the error signal, which is the filter output minus the transmitted signal.

ii **Zero Forcing Equalizer**: approximates the inverse of the channel with a linear filter. The name Zero Forcing corresponds to bringing down the intersymbol interference (ISI) to zero if there is no noise.

When the channel is noisy, ZF receiver will amplify the noise greatly at frequencies where the channel response has a small magnitude in the attempt to invert the channel completely, in which case MMSE is more balanced, which does not usually eliminate ISI completely but instead minimizes the total power of the noise and ISI components in the output. In high Signal-Noise ratio case, MMSE's performance is close to ZF.

1.1.5 Bit Error Rate

After detection, symbol error can be calculated with difference between received symbols and original symbols. Further more, bit error can also be calculated. The bit error rate(BER) is the number of bit errors divided by the total number of transmitted bits during an observed time interval.

The bit error probability P_e is the expected value of the BER. The BER can be considered as an approximate estimate of the bit error probability. This estimate is more accurate for a longer time interval and a larger numbers of experiments. In a communication system, BER may be affected by transmission channel noise, interference, distortion, bit synchronization problems, attenuation, wireless multipath fading, etc.

The BER may be improved by increasing the signal strength, or by applying precoding schemes such as space-time block coding.

1.2 Contribution from Previous Research

Among all the performance measurements of communication systems, the following two are commonly used: capacity and reliability. These two factors govern the increase of data rate and the decrease of probability of error respectively. More specifically, full symbol rate is achieved when one symbol is transmitted by each of the multiple transmitter antennas per time slot (often called per channel use). Papers by Gerard J. Foschini and Michael J. Gans [8], Foschini [3] and Emre Telatar [9] have shown that the channel capacity (a theoretical upper bound on system throughput) for a MIMO system is increased as the number of antennas is increased, proportional to the minimum number of transmitter and receiver antennas. Full diversity [10] is achieved when the total degree of freedom (number of transmitter antennas $\times$ number of receiver antennas) offered in the multiantenna system is utilized. This will ensure a good performance in terms of probability of error for detecting the transmitted symbols at high signal-to-noise ratio (SNR) when a maximum-likelihood (ML) detector is employed [5].

Space-time codes have been developed that simultaneously provide both full diversity and full rate [11], [12] and therefore have fully exploited the advantages of MIMO systems. The good error performance achieved in these designs, however, depends on the ML detector. Transmitters designed to minimize the mean square error of the equalized symbols for both zero-forcing and minimum mean square error (MMSE) equalization were derived in [13]. Design of minimum BER linear precoders for systems with zero-forcing equalization and threshold detection has been studied in [1]. [14] devises a minimum bit error rate (BER) block-based precoder used in block transmission systems with the proposed cascaded zero-forcing (ZF) equalizer. However, they all assumed CSI is fully known by both transmitters and receivers. In [15], when CSI is only known by receiver and totally unknown in transmitter, optimum STBC which not only minimizes BER but also achieves full data rate has been designed using MMSE receivers. However, the channel is supposed to be independent and identically distributed .

1.3 Motivation and Contribution of Thesis

In this thesis, we consider a coherent MIMO system, emphasizing on the simplicity of implementation at both the code generator and the receiver. Specifically, we consider the transmission of a space-time block code (STBC) that is a linear combination of coding matrices weighted by the information symbols through a receiver-correlated flat-fading channel and received by a linear ZF detector. Our target is the design of a code which, while maintaining full data-transmission rate, minimizes the asymptotic average (over all the random channel coefficients) bit error probability of an ZF detector. To this end, we first ensure that the full data rate of symbols is maintained, and then, based on the BER for 4-QAM signals, we derive the conditions for optimal codes and establish a code structure that minimizes the asymptotic average bit error probability. The resulting optimum code structure requires the individual coding matrices to be mutually orthogonal when vectorized and is related to covariance matrix of correlated channel. The first optimum structural characteristics of the coding matrices is described as trace-orthogonal. We have proposed a new approach to express expected value of random correlated channel. We found that our optimum code structure under statistically known CSI has the similar structure under fully known CSI [1].

1.4 Organization of Thesis

In Chapter 1, some background knowledge is introduced. System model of our MIMO communication system is clearly shown in Chapter 2. The original optimization problem of minimizing BER by choosing coding matrix is stated in Chapter 3. Chapter 4 reveals the whole problem reformulation and theoretical analysis. Chapter 5 takes another approach to find the optimum code structure through diversity analysis and asymptotic formula derivation. Combining the conclusion from Chapter 3 and 4, solution to the optimization problem is elaborated in Chapter 6. Last but not the least, simulation results and conclusion have been shown in Chapters 7 and 8. To facilitate continuity in reading, some complicated derivations have been put into appendix.

Chapter 2

System Model

2.1 Precoding

To ease off the complexity demanded at the receiver, proper design of the precoder at the transmitter has received intensive attention in current digital communications due to its ability to improve the system performance. An important aspect in the design of the precoder for prescribed receivers is the availability of channel state information (CSI) at the transmission and reception ends. When perfect CSI is available at the transmitter, there exist solutions to various precoder design problems [16], including maximization of information rate [17], maximization of SNR [13], minimization of the mean squared error [13] and minimization of the bit error probability for Zero-Forcing [16] [1] and MMSE equalization [18].

It is reasonable to assume that CSI is available at the receiver via training. While having CSI at the transmitter allows for better performance, this may not be possible in practice. Throughout this paper, we will assume that perfect CSI is available at the receiver, and only the first- and second-order statistics of the channel is known at the transmitter.

2.2 Spatial Correlation of Channel

Theoretically, the performance of wireless communication systems can be improved by having multiple antennas at the transmitter and the receiver. The idea is that if the propagation channels h between each pair of transmitter and receiver antennas are statistically independent and identically distributed, then multiple independent channels $\mathbf{H}$ with identical characteristics can be created by precoding and be used for either transmitting multiple data streams or increasing the reliability (in terms of bit error rate). In practice, the channels between different antennas are often correlated and therefore the potential multi-antenna gains may not always be obtainable. This is called spatial correlation.

The existence of spatial correlation has been experimentally validated [19] [20]. Spatial correlation is often said to degrade the performance of multi-antenna systems and put a limit on the number of antennas that can be effectively squeezed into a small device (as a mobile phone).

In a narrowband flat-fading channel with M transmitter antennas and N receiver antennas (MIMO), the propagation channel is modeled as [21]

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{2.1}$$

where $\mathbf{y}$ and $\mathbf{x}$ are the $N \times 1$ receive and $M \times 1$ transmit vectors, respectively. The $N \times 1$ noise vector is denoted by $\mathbf{n}$. The ijth element of the $N \times M$ channel matrix $\mathbf{H}$ describes the channel from the jth transmitter antenna to the ith receiver antenna.

When modeling spatial correlation it is useful to employ the Kronecker model, which means that when the correlation between transmitter antennas and receiver antennas are assumed independent and separable, the channel matrix can be expressed as [20]

$$\mathbf{H} \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Sigma}_T \otimes \mathbf{\Sigma}_R) \tag{2.2}$$

where $\otimes$ denotes the Kronecker product (See A.1) and $\mathcal{CN}(\cdot)$ stands for circular symmetric complex normal distribution (See A.2). $\mathbf{\Sigma}_R$ and $\mathbf{\Sigma}_T$ are the receiver-side spatial correlation matrix and transmitter-side spatial correlation matrix respectively. This model is reasonable when the main scattering appears close to the antenna arrays and has been validated by both outdoor and indoor measurements [19] [20].

2.3 System Model of the Thesis

Consider a MIMO communication system (Fig 2.1) having M transmitter antennas and N receiver antennas ($N \geq M$). The symbol stream to be transmitted in T time slots is given by $\{s_l\}$, $l = 1, \ldots, L$, and is divided into M substreams each having T symbols. Thus, we have $L = MT$ symbols to be transmitted. These symbols are selected from a given constellation with zero mean and unity covariance and complex (i.e., $E[\mathbf{ss}^H] = \mathbf{I}$). Each symbol is processed by an $M \times T$ coding matrix C_l, $l = 1, \ldots, L$. The $M \times T$ linear STBC matrix is given by

$$\mathbf{X} = \Sigma_{l=1}^{L} s_l \mathbf{C}_l \tag{2.3}$$

The total power assigned to all the coding matrices is constrained to a constant

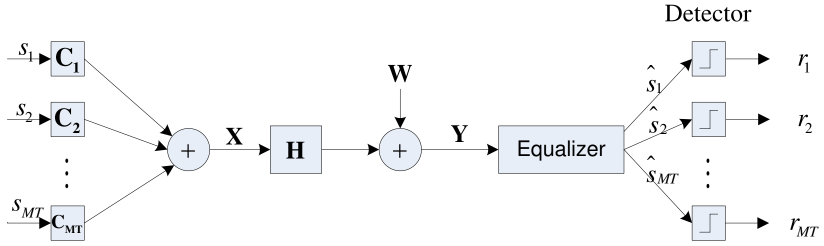


Figure 2.1: MIMO communication system model

L, i.e.,

$$\sum_{l=1}^{L} \operatorname{tr}(\mathbf{C}_l^H \mathbf{C}_l) = L \tag{2.4}$$

The random transmission coefficients h_{nm} are zero-mean, circularly-symmetric complex Gaussian distributed random variable with unit variance. We assume that full knowledge of CSI is available at the receiver. At the transmitter, however, only the first- and second-order statistics of the channels are available. Let the nth row of $\mathbf{H}$ be $\mathbf{h}_n^T = [h_n^1 \cdots h_n^M]$. We assume that [22]

$$E[\mathbf{h}_l \mathbf{h}_n^H] = \begin{cases} \mathbf{\Sigma} & l = n, \\ \mathbf{0} & l \neq n. \end{cases} \tag{2.5}$$

which means channels reaching the same receiver antenna are correlated and channels reaching different receiver antenna are uncorrelated. $\mathbf{H}^H\mathbf{H}$ is of Wishart distribution [23], of N degrees of freedom and with covariance matrix $\mathbf{\Sigma}$, denoted by $\mathcal{W}_M(N, \mathbf{\Sigma})$ (See A.3).

At the receiver, $N \times T$ additive space-time noise matrix $\mathbf{W}$ is of $\mathcal{CN}(\mathbf{0}, \mathbf{I})$ with covariance matrix $\mathbf{R}_{ww} = \mathbf{I}$.

The $N \times T$ received signal Y is

$$\mathbf{Y} = \sqrt{\frac{\rho}{M}}\mathbf{H}\mathbf{X} + \mathbf{W} \tag{2.6}$$

It can be vectorized to:

$$\begin{aligned} \mathbf{y} &= \mathrm{vec}(\mathbf{Y}) \\ &= \sqrt{\frac{\rho}{M}}(\mathbf{I}_T \otimes \mathbf{H})\mathbf{F}\mathbf{s} + \mathbf{w} \\ &= \sqrt{\frac{\rho}{M}}\mathcal{H}\mathbf{s} + \mathbf{w} \end{aligned}$$

where $\mathbf{F} = [\mathrm{vec}(\mathbf{C}_1) \cdots \mathrm{vec}(\mathbf{C}_L)] : MT \times MT$,

$\mathcal{H} = (\mathbf{I}_T \otimes \mathbf{H})\mathbf{F} : NT \times MT$

Now we have vector-form signal at the input of receiver end. In next chapter, zero-forcing equalizers will be used to acquire detected symbols. Asymptotic average bit error rate will be expressed so that our problem can be stated.

Chapter 3

Problem Establishment

3.1 Zero-Forcing Equalization

The Zero-Forcing Equalizer applies the inverse of the channel to the received signal to restore the signal before the channel so that [13]

$$\sqrt{\frac{\rho}{M}}\mathbf{G}\mathcal{H} = \mathbf{I} \tag{3.1}$$

Since we have $N \geq M$, $\mathcal{H}$ $(NT \times MT)$ is a tall matrix, for ZF equalizer $\mathbf{G}$

$$\begin{aligned}\mathbf{G} &= \sqrt{\frac{M}{\rho}}\mathcal{H}^{\dagger} \\ &= \sqrt{\frac{M}{\rho}}\left(\mathcal{H}^{H}\mathcal{H}\right)^{-1}\mathcal{H}^{H}\end{aligned}$$

where $(\cdot)^H$ stands for hermitian (conjugate and transfer) and $(\cdot)^{\dagger}$ stands for pseudo-inverse (See A.4).

By facilitating the linear equalizer $\mathbf{G}$, the $L \times 1$ equalized signal $\hat{\mathbf{s}}$ can be written as

$$\begin{aligned}\hat{\mathbf{s}} &= \mathbf{G}\mathbf{y} \\ &= \sqrt{\frac{\rho}{M}}\mathbf{G}\mathcal{H}\mathbf{s} + \mathbf{G}\mathbf{w} \\ &= \mathbf{s} + \mathbf{G}\mathbf{w}\end{aligned}$$

The error vector $\mathbf{e} \triangleq (\mathbf{s} - \hat{\mathbf{s}})$ is given by

$$\mathbf{e} = \mathbf{G}\mathbf{w} \tag{3.2}$$

where $\mathbf{w}$ is assumed white so that $\mathbf{R}_{ww} = \mathbf{I}$.

So that the covariance matrix, $\mathbf{V}_{se}$, of the error vector is given by

$$\begin{aligned}\mathbf{V}_{se} &= \mathrm{E}[\mathbf{e}\mathbf{e}^H] \\ &= \mathbf{G}\mathbf{R}_{ww}\mathbf{G}^H \\ &= \frac{M}{\rho}\left(\mathcal{H}^H\mathcal{H}\right)^{-1}\end{aligned} \tag{3.3}$$

where $(\cdot)_{se}$ stands for symbol error, and the bit error rate will be expressed in the following section.

3.2 Asymptotic Average Bit Error Rate

For 4-QAM signals, each transmitted symbol consists of one bit in each of its real and imaginary parts. To evaluate bit error rate, we must then separate the transmitted and the detected 4-QAM symbols into their respective real and imaginary parts and examine the error in both parts.

Writing $\boldsymbol{\sigma} = [\mathbf{s}_{Re}\ \mathbf{s}_{Im}]^T$ and $\hat{\boldsymbol{\sigma}} = [\hat{\mathbf{s}}_{Re}\ \hat{\mathbf{s}}_{Im}]^T$ which are $2L \times 1$ vectors with $(\cdot)_{Re}$ and $(\cdot)_{Im}$ denoting respectively the real and imaginary parts of a complex quantity, the covariance matrix of bit error $\mathbf{V}_{be}$ is determined by

$$\begin{aligned} \mathbf{V}_{be} &= \mathrm{E}[(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}})(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}})^H] \\ &= \frac{M}{\rho}[\mathbf{T}^H\hat{\mathbf{F}}^H\hat{\mathbf{H}}^H\hat{\mathbf{H}}\hat{\mathbf{F}}\mathbf{T}]^{-1} \end{aligned} \tag{3.4}$$

where $2L \times 2L$ matrices $\hat{\mathbf{H}}, \hat{\mathbf{F}}, \mathbf{T}$ are defined as

$$\hat{\mathbf{H}} = \begin{pmatrix} \mathbf{I}_T \otimes \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_T \otimes \mathbf{H}^* \end{pmatrix}$$

$$\hat{\mathbf{F}} = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^* \end{pmatrix}$$

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I}_L & j\mathbf{I}_L \\ \mathbf{I}_L & -j\mathbf{I}_L \end{pmatrix}$$

where $(\cdot)^*$ stands for conjugate.

The lth element of $\mathbf{V}_{be}$, denoted by $[\mathbf{V}_{be}]_{ll}$, $l = 1, \ldots, 2L$, is the mean square error(MSE) of the lth bit of $\boldsymbol{\sigma}$.

According to [24], the bit error rate (BER) in the detection of l th symbol of $\mathbf{s}$ is given by

$$\mathcal{P}_{el} = \mathrm{E}_{\mathrm{H}}\left\{Q\left(\sqrt{\gamma_l}\right)\right\} \tag{3.5}$$

where γ_l is the signal to interference plus noise ratio (SINR) associated with lth bit of the equalized signal vector $\hat{\boldsymbol{\sigma}}$ and $\mathrm{E_H}(\cdot)$ is the expectation taken over all the random channel matrices $\mathbf{H}$, and $Q(\cdot)$ stands for Q-function (See A.5).

For our system in which a ZF equalizer is employed, because there is no inter-symbol interference, the SINR in the lth bit, which equals to signal to noise ratio(SNR), can be expressed in terms of MSE [13] such that

$$\gamma_l = \frac{1}{[\mathbf{V}_{be}]_{ll}} \tag{3.6}$$

Substituting Eq. (3.6) into Eq. (F.76), the average probability of error for lth bit can be written as

$$\mathcal{P}_{el} = \mathrm{E_H}\left\{ Q\left(\sqrt{[\mathbf{V}_{be}]_{ll}^{-1}} \right) \right\} \tag{3.7}$$

The averaged bit error probability over all the $2L$ bits is:

$$\mathcal{P}_e(\mathbf{F}) = \frac{1}{2L}\sum_{l=1}^{2L} \mathrm{E_H}\left\{ Q\left(\sqrt{[\mathbf{V}_{be}]_{ll}^{-1}} \right) \right\} \tag{3.8}$$

Eq. (3.8) yields an expression for the asymptotic average bit error probability of the MIMO system that transmits a 4-QAM signal and is equipped with a ZF receiver. It is a function of $[\mathbf{V}_{be}]_{ll}$ which, in turn, is a function of $\mathbf{F}$.

3.3 Problem Statement

Since our goal is to design coding matrix $\mathbf{F}$ to minimize the average asymptotic BER given by Eq. (4.1), the optimization problem can be stated as

$$\begin{aligned} &\min_{\mathbf{F}} \mathcal{P}_e(\mathbf{F}) \\ &\text{s.t. } \operatorname{tr}(\mathbf{F}^H\mathbf{F}) = L \end{aligned} \tag{3.9}$$

This problem is solved in following two stages:

i Minimize the lower bound of BER

ii Find the optimum code structure to achieve this lower bound

In the next chapter, we will show how to find the optimum solution following our methodology.

Chapter 4

Problem Reformulation

4.1 Convex Optimization Problems

4.1.1 Convex Sets

Line segment is defined as all points x between x_1 and x_2 such as

$$x = \theta x_1 + (1 - \theta)x_2$$

with $0 \leq \theta \leq 1$.

A **convex set** is a set C which contains **line segment** between any two points in the set such as

$$x_1, x_2 \in C,\ 0 \leq \theta \leq 1 \Longrightarrow \theta x_1 + (1 - \theta)x_2 \in C$$

Based on the definition of convex set the concepts of affine functions and convex functions are introduced.

4.1.2 Affine Functions

A function f: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is **affine** if it is a sum of a linear function and a constant, i.e., if it has the form $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. $\mathbb{R}^n$ stands for n-dimension real number set. $\mathbb{R}^{m \times n}$ stands for $m \times n$-dimension real number set. **dom** f represents the domain of a function, which is the set of 'input" or argument values for which the function is defined. Here, **dom** f is the set $\mathbb{R}^n$.

4.1.3 Convex Functions

f: $\mathbb{R}^n \rightarrow \mathbb{R}$ is strictly convex if **dom** f is a convex set and

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) < \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})$$

for all $\mathbf{x}$, $\mathbf{y} \in$ **dom** f, $0 < \theta < 1$.

After showing affine functions and convex functions, convex optimization problems can be defined as follows.

4.1.4 Convex Optimization Problems

A convex optimization problem is one of the form [25]

$$\begin{aligned} &\text{minimize} && f_0(\mathbf{x}) \\ &\text{subject to} && f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & && \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in \mathbf{R}^{m \times n},\ \mathbf{b} \in \mathbf{R}^m \end{aligned}$$

where f_0, ..., f_m are convex functions.

The convex problem has three additional requirements:

- the objective function $f_0(\mathbf{x})$ must be convex,
- the inequality constraint functions $f_i(\mathbf{x})$ must be convex, and
- the equality constraint functions $\mathbf{A}\mathbf{x} = \mathbf{b}$ must be affine.

If the objective is strictly convex, then the optimal set contains at most one point. For convex problem the local optimal set is also the global optimal set, which can help us reduce the range of variables and the complexity. In our thesis, we will first convert our non-convex problem to a convex problem by relaxation and then solve the reformulated problem in following sections.

4.2 Minimize Lower Bound

From Chapter 3, we derive the averaged bit error probability

$$\mathcal{P}_e(\mathbf{F}) = \frac{1}{2L} \sum_{l=1}^{2L} \mathrm{E}_{\mathrm{H}} \left\{ Q\left(\sqrt{[\mathbf{V}_{be}]_{ll}^{-1}} \right) \right\} \tag{4.1}$$

This is the objective of our problem, and first we find the range of variable in which our objective is a convex function.

The function $f(x) = Q(\sqrt{x^{-1}})$ is convex in the interval $0 < x \leq \frac{1}{3}$ (See B.1.3).

Therefore, the function inside the braces in Eq. (4.1) is convex when $0 < [\mathbf{V}_{be}]_{ll} \leq$

$\frac{1}{3}$, $l = 1, \ldots, 2L$, applying Jensen's inequality on our convex function, we obtain

$$\begin{aligned}\mathcal{P}_e(\mathbf{F}) &\geq \mathrm{E}_{\mathrm{H}}\left\{Q\left(\sqrt{\frac{1}{\frac{1}{2L}\sum_{l=1}^{2L}[\mathbf{V}_{be}]_{ll}}}\right)\right\} \\ &= \mathrm{E}_{\mathrm{H}}\left\{Q\left(\sqrt{\frac{2L}{\mathrm{tr}(\mathbf{V}_{be})}}\right)\right\} \\ &= \mathrm{E}_{\mathrm{H}}\left\{Q\left(\sqrt{\frac{L}{\mathrm{tr}(\mathbf{V}_{se})}}\right)\right\} \end{aligned} \tag{4.2}$$

which represents a lower bound for the average asymptotic BER.

Equality holds if and only if the following condition is satisfied

$$[\mathbf{V}_{be}]_{ii} = [\mathbf{V}_{be}]_{jj}, \ \forall i, j = 1, \ldots, 2L. \tag{4.3}$$

which is the first condition to minimize the lower bound.

Now our job is to minimize the lower bound in Eq. (4.2). Since function $f(x) = Q(\sqrt{x^{-1}})$ is monotonically increasing with x, to minimize $\mathcal{P}_e(\mathbf{F})$ is the same as to minimize $\mathrm{tr}(\mathbf{V}_{se})$ in Eq. (4.2).

First we try to express $\mathrm{tr}(\mathbf{V}_{se})$ by substituting $\mathcal{H} = (\mathbf{I}_T \otimes \mathbf{H})\mathbf{F}$ into Eq. (3.3) and then apply the two properties of matrix computation $\mathrm{tr}(\mathbf{XY}) = \mathrm{tr}(\mathbf{XY})$ and

$(\mathbf{Z} \otimes \mathbf{X}) \cdot (\mathbf{Z} \otimes \mathbf{Y}) = \mathbf{Z} \otimes \mathbf{XY}$:

$$\begin{aligned}
\mathrm{tr}(\mathbf{V}_{se}) =& \mathrm{tr}\left(\frac{M}{\rho}(\mathcal{H}^H\mathcal{H})^{-1}\right) \\
&= \mathrm{tr}\left(\frac{M}{\rho}(\mathcal{H}\mathcal{H}^H)^{-1}\right) \\
&= \mathrm{tr}\left(\frac{M}{\rho}\left((\mathbf{I}_T \otimes \mathbf{H})\mathbf{F}\mathbf{F}^H(\mathbf{I}_T \otimes \mathbf{H}^H)\right)^{-1}\right) \\
&= \mathrm{tr}\left(\frac{M}{\rho}\left((\mathbf{I}_T \otimes \mathbf{H}^H\mathbf{H})\mathbf{F}\mathbf{F}^H\right)^{-1}\right) \\
&= \mathrm{tr}\left(\frac{M}{\rho}\left((\mathbf{I}_T \otimes (\mathbf{H}^H\mathbf{H})^{\frac{1}{2}})\mathbf{F}\mathbf{F}^H(\mathbf{I}_T \otimes (\mathbf{H}^H\mathbf{H})^{\frac{1}{2}})\right)^{-1}\right) \\
&= \mathrm{tr}\left(\frac{M}{\rho}\left((\mathbf{I}_T \otimes \mathbf{\Phi})\,\mathbf{A}\,(\mathbf{I}_T \otimes \mathbf{\Phi})\right)^{-1}\right)
\end{aligned} \tag{4.4}$$

where $\mathbf{\Phi} = (\mathbf{H}^H\mathbf{H})^{\frac{1}{2}}$: $M \times M$ and $\mathbf{A} = \mathbf{F}\mathbf{F}^H$: $MT \times MT$.

To facilitate the development of the proof, we first introduce the following lemma on the trace of a matrix [26].

Lemma: For any nonsingular Hermitian symmetric positive semidefinite matrix $\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{pmatrix}$ we have

$$\mathrm{tr}(\mathbf{Z}^{-1}) \geq \mathrm{tr}(\mathbf{Z}_{11}^{-1}) + \mathrm{tr}(\mathbf{Z}_{22}^{-1}) \tag{4.5}$$

where equality holds if and only if $\mathbf{Z}_{12} = 0$, i.e., if and only if $\mathbf{Z}$ is block diagonal.

Applying the lemma to Eq. (4.4), we have

$$\mathrm{tr}(\mathbf{V}_{se}) \geq \sum_{t=1}^{T} \mathrm{tr}\left(\frac{M}{\rho}(\mathbf{\Phi}\mathbf{A}_{tt}\mathbf{\Phi}^H)^{-1}\right) \tag{4.6}$$

where $\mathbf{A}_{tt}$, $t = 1, \ldots, T$, are $M \times M$ matrices on the diagonal of $\mathbf{A}$. Now we can see that the matrices have been simplified.

Equality holds if and only if $\mathbf{A}$ is block diagonal, i.e.,

$$\mathbf{A}_{ij} = \mathbf{0} \text{ for } i, j = 1, \ \ldots, \ T, \ i \neq j. \tag{4.7}$$

Also, since $\mathrm{tr}(\mathbf{Z}^{-1})$ is convex with positive semidefinite matrix $\mathbf{Z}$ [25], applying Jensen's inequality (See B.2), we have

$$\begin{aligned}
\mathrm{tr}(\mathbf{V}_{se}) &= T \cdot \frac{1}{T} \sum_{t=1}^{T} \mathrm{tr}\left(\frac{M}{\rho} (\mathbf{\Phi} \mathbf{A}_{tt} \mathbf{\Phi}^H)^{-1} \right) \\
&\geq T \cdot \mathrm{tr}\left(\frac{M}{\rho} (\mathbf{\Phi} \frac{1}{T} \sum_{t=1}^{T} \mathbf{A}_{tt} \mathbf{\Phi}^H)^{-1} \right) \\
&= T \cdot \mathrm{tr}\left(\frac{M}{\rho} (\mathbf{\Phi} \bar{\mathbf{A}} \mathbf{\Phi}^H)^{-1} \right)
\end{aligned}$$

where $\bar{\mathbf{A}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{A}_{tt}$.

Equality holds if and only if all $\mathbf{A}_{tt}$ are equal, i.e.,

$$\mathbf{A}_{tt} = \bar{\mathbf{A}} \text{ for } t = 1, \ \ldots, \ T. \tag{4.8}$$

Above all, $\mathbf{F}\mathbf{F}^H = \mathbf{A}$ is block diagonal and with each matrix on the diagonal is $\bar{\mathbf{A}}$, which is the second condition to minimize the lower bound. If we define $\bar{\mathbf{A}} = \bar{\mathbf{F}}\bar{\mathbf{F}}^H$, the optimum code we design has the structure (See C.8)

$$\mathbf{F} = \begin{pmatrix} \bar{\mathbf{F}} & 0 & \cdots & 0 \\ 0 & \bar{\mathbf{F}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \bar{\mathbf{F}} \end{pmatrix} \mathbf{V}_F^H \tag{4.9}$$

where $\mathbf{V}_F^H$ is a unitary matrix which can be generated later.

Now the problem to find the optimum $\mathbf{F}$ is simplified to find the optimum $\bar{\mathbf{F}}$ and $\mathbf{V}_F^H$.

Therefore, substituting the lower bound of $\mathrm{tr}(\mathbf{V}_{se})$ into Eq. (4.2), we have

$$\mathcal{P}_e(\mathbf{F}) \geq \mathrm{E}_{\mathrm{H}} \left\{ Q \left(\sqrt{\frac{MT}{T\mathrm{tr}\left(\frac{M}{\rho}(\mathbf{\Phi}\bar{\mathbf{A}}\mathbf{\Phi}^H)^{-1}\right)}} \right) \right\} \tag{4.10}$$

$$= \mathrm{E}_{\mathrm{H}} \left\{ Q \left(\sqrt{\frac{\rho}{\mathrm{tr}\left((\mathbf{\Phi}\bar{\mathbf{A}}\mathbf{\Phi}^H)^{-1}\right)}} \right) \right\} \tag{4.11}$$

Equality in (4.10) holds when Eqs. (4.3), (4.7) and (4.8) are met simultaneously.

Next, we will simplify the matrix again to change the original matrix variable into a single variable.

In Eq. (4.11), if we define the trace in the denominator as x, substitute $\bar{\mathbf{A}} = \bar{\mathbf{F}}\bar{\mathbf{F}}^H$

and $\mathbf{\Phi} = (\mathbf{H}^H\mathbf{H})^{\frac{1}{2}}$ into x, we obtain

$$\begin{aligned} x &= \operatorname{tr}\left(\left(\mathbf{\Phi}\bar{\mathbf{A}}\mathbf{\Phi}^H\right)^{-1}\right) \qquad (4.12) \\ &= \operatorname{tr}\left(\left(\mathbf{H}^H\mathbf{H}\bar{\mathbf{F}}\bar{\mathbf{F}}^H\right)^{-1}\right) \\ &= \operatorname{tr}\left(\left(\left(\mathbf{H}\bar{\mathbf{F}}\right)^H\left(\mathbf{H}\bar{\mathbf{F}}\right)\right)^{-1}\right) \qquad (4.13) \end{aligned}$$

In our assumption, $\mathbf{H}$ is a correlated channel, to simplify the problem, we will convert this correlated channel to an i.i.d (independent and identically distributed) channel $\tilde{\mathbf{H}} = \mathbf{H}\mathbf{\Sigma}^{-\frac{1}{2}}$ (See A.3.3).

Besides, we define

$$\tilde{\mathbf{F}} = \mathbf{\Sigma}^{\frac{1}{2}}\bar{\mathbf{F}} \qquad (4.14)$$

so that

$$\begin{aligned} \tilde{\mathbf{H}}\tilde{\mathbf{F}} &= \mathbf{H}\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{\Sigma}^{\frac{1}{2}}\bar{\mathbf{F}} \\ &= \mathbf{H}\bar{\mathbf{F}} \qquad (4.15) \end{aligned}$$

Substituting Eq. (4.15) into Eq. (4.12), we have

$$\begin{aligned} x &= \mathrm{tr}\left(\left((\mathbf{H}\bar{\mathbf{F}})^H(\mathbf{H}\bar{\mathbf{F}})\right)^{-1}\right) \\ &= \mathrm{tr}\left(\left(\left(\tilde{\mathbf{H}}\tilde{\mathbf{F}}\right)^H\left(\tilde{\mathbf{H}}\tilde{\mathbf{F}}\right)\right)^{-1}\right) \\ &= \mathrm{tr}\left(\left(\tilde{\mathbf{F}}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\tilde{\mathbf{F}}\right)^{-1}\right) \\ &= \mathrm{tr}\left(\left((\tilde{\mathbf{H}}^H\tilde{\mathbf{H}})(\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H)\right)^{-1}\right) \end{aligned} \quad (4.16)$$

So until now, the problem to find the optimum $\bar{\mathbf{F}}$ becomes to find the optimum $\tilde{\mathbf{F}}$.

We try to simplify the matrix to single variable which can make the optimization problem easier, so that we use eigendecomposition on symmetric matrix $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H$ here:

$$\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H = \mathbf{V}_{\tilde{F}}\mathbf{D}_{\tilde{F}}\mathbf{V}_{\tilde{F}}^H \quad (4.17)$$

where $\mathbf{D}_{\tilde{F}}$ is diagonal matrix with each element on the diagonal $d_i \geq 0$, $i = 1, \ldots, M$, which is also one of the eigenvalues of $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H$.

Substitute the eigendecomposition of $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H$ into x in Eq. (4.16), we have

$$\begin{aligned} x &= \mathrm{tr}\left(\left((\tilde{\mathbf{H}}^H\tilde{\mathbf{H}})(\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H)\right)^{-1}\right) \\ &= \mathrm{tr}\left(\left((\tilde{\mathbf{H}}^H\tilde{\mathbf{H}})(\mathbf{V}_{\tilde{F}}\mathbf{D}_{\tilde{F}}\mathbf{V}_{\tilde{F}}^H)\right)^{-1}\right) \\ &= \mathrm{tr}\left(\left(\mathbf{V}_{\tilde{F}}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{V}_{\tilde{F}}\mathbf{D}_{\tilde{F}}\right)^{-1}\right) \\ &= \mathrm{tr}\left(\left((\tilde{\mathbf{H}}\mathbf{V}_{\tilde{F}})^H(\tilde{\mathbf{H}}\mathbf{V}_{\tilde{F}})\mathbf{D}_{\tilde{F}}\right)^{-1}\right) \\ &= \mathrm{tr}\left(\left(\hat{\mathbf{H}}^H\hat{\mathbf{H}}\mathbf{D}_{\tilde{F}}\right)^{-1}\right) \\ &= \sum_{i=1}^{M}[(\hat{\mathbf{H}}^H\hat{\mathbf{H}})^{-1}]_{ii}d_i^{-1} \end{aligned} \tag{4.18}$$

where $\hat{\mathbf{H}} = \tilde{\mathbf{H}}\mathbf{V}_{\tilde{F}}$, and $\mathbf{V}_{\tilde{F}}$ is a $M \times M$ unitary matrix. Applying the property of Wishart distribution again, we have $\hat{\mathbf{H}}^H\hat{\mathbf{H}} \sim \mathcal{W}(N, \mathbf{I}_M)$ (See A.3.3), so that $\hat{\mathbf{H}}$ is also an i.i.d channel now.

Substituting x into Eq. (4.11), and also applying the definition of expectation, we have

$$\begin{aligned} \mathcal{P}_e(\mathbf{F}) &\geq \mathrm{E_H}\left\{Q\left(\sqrt{\frac{\rho}{\mathrm{tr}\left((\mathbf{\Phi}\bar{\mathbf{A}}\mathbf{\Phi}^H)^{-1}\right)}}\right)\right\} \\ &= \mathrm{E_H}\left\{Q\left(\sqrt{\frac{\rho}{x}}\right)\right\} \\ &= \int_H p(H)Q\left(\sqrt{\frac{\rho}{x}}\right)dH \\ &= \int_x p(x)Q\left(\sqrt{\frac{\rho}{x}}\right)dx \\ &= \mathrm{E}_x\left\{Q\left(\sqrt{\frac{\rho}{x}}\right)\right\} \end{aligned} \tag{4.19}$$

In Section 3.2, we already know that function $f(x) = Q(\sqrt{x^{-1}})$ is convex in the interval $0 < x \leq \frac{1}{3}$, so that $Q(\sqrt{\rho x^{-1}})$ is convex in the interval $0 < x \leq \frac{\rho}{3}$. Now the range of x is known, in the next section, we will develop a new method to express probability density function of x: $p(x)$.

4.3 Probability Distribution of Variable x

To simplify the problem to calculate probability density function of x, we assume the number of transmitter antennas $M = 2$ and the number of receiver antennas $N = 2$, which means $\hat{\mathbf{H}}$ is a 2×2 matrix. Applying the QR decomposition (See C.1) on $\hat{\mathbf{H}}$, we have

$$\begin{aligned}\hat{\mathbf{H}} &= \mathbf{Q}_{\hat{H}}\mathbf{R}_{\hat{H}} \\ &= \mathbf{Q}_{\hat{H}} \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}\end{aligned}$$

where $\mathbf{Q}_{\hat{H}}$ is a unitary matrix meaning $\mathbf{Q}_{\hat{H}}^H \mathbf{Q}_{\hat{H}} = \mathbf{I}_2$ and $\mathbf{R}_{\hat{H}}$ is an upper triangular matrix with elements r_{11}, r_{12}, r_{22}.

Therefore,

$$\begin{aligned}
\hat{\mathbf{H}}^H\hat{\mathbf{H}} &= \mathbf{R}_{\hat{H}}^H\mathbf{Q}_{\hat{H}}^H\mathbf{Q}_{\hat{H}}\mathbf{R}_{\hat{H}} \\
&= \mathbf{R}_{\hat{H}}^H\mathbf{R}_{\hat{H}} \\
&= \begin{pmatrix} r_{11}^* & 0 \\ r_{12}^* & r_{22}^* \end{pmatrix}\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \\
&= \begin{pmatrix} |r_{11}|^2 & r_{11}^* r_{12} \\ r_{11}r_{12}^* & |r_{12}|^2 + |r_{22}|^2 \end{pmatrix}
\end{aligned} \tag{4.20}$$

If we define $x_1 = |r_{11}|^2 \geq 0$, $x_2 = |r_{22}|^2 \geq 0$ and $x_3 = |r_{12}|^2 \geq 0$, applying the Bartlett decomposition [27], we have (See C.3),

$$x_1 = |r_{11}|^2 \sim \chi_4^2$$

$$x_2 = |r_{22}|^2 \sim \chi_2^2$$

$$x_3 = |r_{12}|^2 \sim \chi_2^2$$

where χ_4^2 and χ_2^2 are chi-square distributions (See C.2), applying the definition of chi-square distribution, we can obtain the probability density function of x_1, x_2 and x_3 respectively:

$$p(x_1) = x_1 e^{-x_1}$$

$$p(x_2) = e^{-x_2}$$

$$p(x_3) = e^{-x_3}$$

From Eq. (4.20), the determinant of matrix $\hat{\mathbf{H}}^H\hat{\mathbf{H}}$ is

$$\begin{aligned}\det(\hat{\mathbf{H}}^H\hat{\mathbf{H}}) &= |r_{11}|^2(|r_{12}|^2+|r_{22}|^2) - r_{11}^* r_{12} r_{11} r_{12}^* \\ &= |r_{11}|^2|r_{12}|^2 + |r_{11}|^2|r_{22}|^2 - |r_{11}|^2|r_{12}|^2 \\ &= |r_{11}|^2|r_{22}|^2\end{aligned}$$

So that the inverse of matrix $\hat{\mathbf{H}}^H\hat{\mathbf{H}}$ is

$$\begin{aligned}\left(\hat{\mathbf{H}}^H\hat{\mathbf{H}}\right)^{-1} &= \frac{1}{\det(\hat{\mathbf{H}}^H\hat{\mathbf{H}})}\begin{pmatrix} |r_{12}|^2+|r_{22}|^2 & -r_{11}^* r_{12} \\ -r_{11} r_{12}^* & |r_{11}|^2 \end{pmatrix} \\ &= \frac{1}{|r_{11}|^2|r_{22}|^2}\begin{pmatrix} |r_{12}|^2+|r_{22}|^2 & -r_{11}^* r_{12} \\ -r_{11} r_{12}^* & |r_{11}|^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{|r_{12}|^2+|r_{22}|^2}{|r_{11}|^2|r_{22}|^2} & \frac{-r_{11}^* r_{12}}{|r_{11}|^2|r_{22}|^2} \\ \frac{-r_{11} r_{12}^*}{|r_{11}|^2|r_{22}|^2} & \frac{|r_{11}|^2}{|r_{11}|^2|r_{22}|^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{|r_{12}|^2+|r_{22}|^2}{|r_{11}|^2|r_{22}|^2} & \frac{-1}{r_{11} r_{12}^*} \\ \frac{-1}{r_{11}^* r_{12}} & \frac{1}{|r_{22}|^2} \end{pmatrix}\end{aligned}$$

Therefore, substitute this into x in Eq. (4.18), we obtain

$$\begin{aligned}x &= \sum_{i=1}^{M}[(\hat{\mathbf{H}}^H\hat{\mathbf{H}})^{-1}]_{ii} d_i^{-1} \\ &= \left(\frac{|r_{22}|^2+|r_{12}|^2}{|r_{11}|^2|r_{22}|^2}\right) d_1^{-1} + \frac{1}{|r_{22}|^2} d_2^{-1} \\ &= \left(\frac{x_2+x_3}{x_1 x_2}\right) d_1^{-1} + \frac{1}{x_2} d_2^{-1}\end{aligned}$$

Because r_{11}, r_{12} and r_{22} are independent of each other, x_1, x_2 and x_3 are also independent of each other. So that we can write the probability distribution function of x as

$$\begin{aligned} F_x(t) &= P\{x \le t\} \\ &= P\left(\left(\frac{x_2+x_3}{x_1x_2}\right)d_1^{-1} + \frac{d_2^{-1}}{x_2} \le t\right) \\ &= P\left(x_3 \le \frac{x_1x_2t - x_2d_1^{-1} - x_1d_2^{-1}}{d_1^{-1}}\right) \\ &= \int_{x_1}\int_{x_2}\int_{x_3} p(x_1)p(x_2)p(x_3)dx_3dx_2dx_1 \end{aligned} \tag{4.21}$$

After calculation of the integrals (See Appendix D), we can express the distribution function $F_x(t)$ as

$$F_x(t) = \int_0^\infty \frac{u}{t^2}e^{-\frac{(u+d_1^{-1})(u+d_2^{-1})}{tu}}du \tag{4.22}$$

where $u = x_1t - d_1^{-1}$

We can see that the distribution function changes according to different values of d_1 and d_2, Figure 4.1 shows the distribution function $F_x(t)$ in the case $d_1 = 1$ and $d_2 = 0.6$, 0.8, 1 and 1.2.

$F_x'(t)$, w.r.t the first-order derivative of $F_x(t)$, is the probability density function of x,

$$\begin{aligned} p(x) &= F_x'(t) \\ &= \int_0^\infty \left[\frac{-2u}{t^3}e^{-\frac{(u+d_1^{-1})(u+d_2^{-1})}{tu}} + \frac{u}{t^2}\frac{(u+d_1^{-1})(u+d_2^{-1})}{t^2u}e^{-\frac{(u+d_1^{-1})(u+d_2^{-1})}{tu}}\right]du \quad (4.23) \\ &= \int_0^\infty e^{-\frac{(u+d_1^{-1})(u+d_2^{-1})}{tu}}\left(\frac{-2u}{t^3} + \frac{(u+d_1^{-1})(u+d_2^{-1})}{t^4}\right)du \quad (4.24) \end{aligned}$$

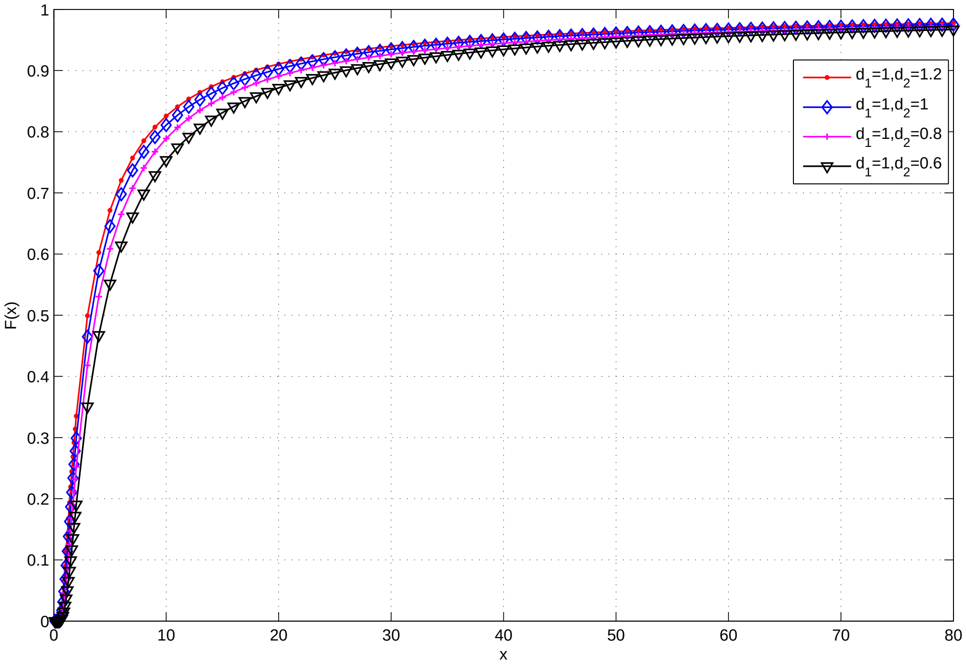


Figure 4.1: Distribution function of x when $d_1 = 1$ and $d_2 = 0.6$, 0.8, 1 and 1.2

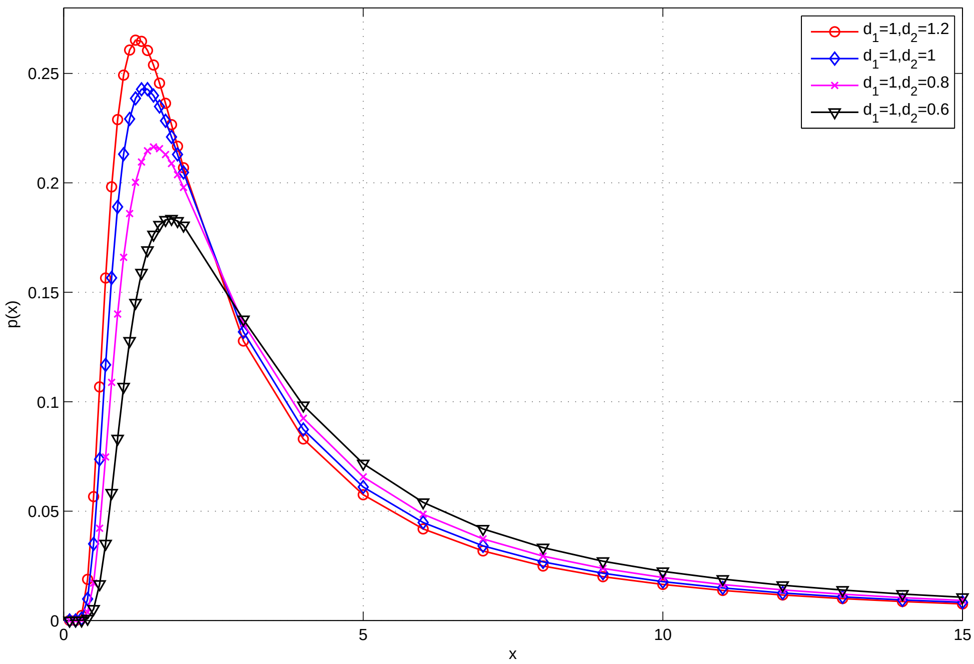


Figure 4.2: Probability density function of x when $d_1 = 1$ and $d_2 = 0.6$, 0.8, 1 and 1.2

Although the formula of $p(x)$ looks complicated, we have checked that the integral of $p(x)$ approximates to one, which means the result is reasonable.

Figure 4.2 shows the probability density function $p(x)$ in the case $d_1 = 1$ and $d_2 = 0.6$, 0.8, 1 and 1.2.

For now we have solved the problem of random correlated channel. In the next section, we will express the formula of lower bound of asymptotic average bit error rate $P_e(\mathbf{F})$.

4.4 Reformulated Optimization Problem

As described before, to meet the requirement of convex function, we have $0 < x \leq \frac{\rho}{3}$. Substituting probability density function $p(x)$ and trigonometric form of Q-function (See A.5) into Eq. (4.19), we have

$$\begin{aligned} P_e(\mathbf{F}) &\geq \mathrm{E}_x \left\{ Q\left(\sqrt{\frac{\rho}{x}}\right) \right\} \\ &= \int_0^{\frac{\rho}{3}} p(x) Q\left(\sqrt{\frac{\rho}{x}}\right) dx \\ &= \int_0^{\frac{\rho}{3}} p(x) \left[\frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{\left(-\frac{\rho}{2xsin^2\theta}\right)} d\theta \right] dx \end{aligned} \tag{4.25}$$

We can see that there is another integral within $p(x)$, so after complicated calculation of integrals, eventually we have following form of $P_e(\mathbf{F})$ (See Appendix E):

$$P_e(\mathbf{F}) \geq \int_0^{\infty} \frac{u}{\rho^2} \left[\frac{9}{2} e^{-\frac{3}{2}(a-1)} \mathrm{erfc}(\frac{\sqrt{6}}{2}) - \sqrt{\frac{27}{2\pi}} (a^{-2} + a^{-1}) e^{-\frac{3}{2}a} - \frac{3}{2} a^{-\frac{5}{2}} \mathrm{erfc}(\frac{\sqrt{6a}}{2}) \right] du \tag{4.26}$$

where $a = \frac{2(u+d_1^{-1})(u+d_2^{-1})}{\rho u} + 1$ and error function $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$. From eigen-decomposition of $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H$ in Eq. (4.16), d_1 and d_2 are two eigenvalues of $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H$.

Therefore, the original optimization problem in Eq. (3.9) is reformulated as

$$\begin{aligned} &\min_{d_1, d_2} \mathcal{P}_e(\mathbf{F}) \\ &\text{s.t. } \mathrm{tr}(\mathbf{F}^H \mathbf{F}) = L \end{aligned} \tag{4.27}$$

In Chapter 5, we will solve this reformulated problem by minimizing the asymptotic formula of averaged bit error rate in Eq. (4.26).

Chapter 5

Diversity Analysis and Asymptotic Formula

5.1 Diversity

In telecommunications, a diversity scheme refers to a method for improving the reliability of a message signal by using two or more communication channels with different characteristics. Diversity plays an important role in combatting fading and co-channel interference. It is based on the fact that individual channels experience different levels of fading and interference. Multiple versions of the same signal may be transmitted and/or received and combined in the receiver.

Space diversity is one kind of diversity. The signal is transmitted over several different propagation paths which can be achieved by antenna diversity using multiple transmitter antennas and/or multiple receiving antennas. Diversity techniques may exploit the multipath propagation, resulting in a diversity gain.

5.1.1 Diversity Gain

Diversity gain, d, is defined as [28]

$$d = -\lim_{\rho \to \infty} \log P_e / \log \rho \tag{5.1}$$

where P_e is the detection error probability and ρ is the signal to noise ratio (SNR). Diversity gain is a concept central to code designs since it is an indication of the rate of decay of the error probability with SNR when the SNR is high. Thus, a system that fully utilizes the diversity advantage of MIMO channels is superior in error rate performance at high SNR to those which do not have full diversity.

The minimum BER at high SNR can be written in asymptotic formula form in terms of diversity gain d defined in Eq. (5.1). We concentrate on the dominant term (the term containing the lowest order of ρ^{-1}) in the asymptotic performance at high SNR, i.e.,

$$P_e = \mathcal{K}^{-1}\rho^{-d} + (\text{terms involving higher order of } \rho^{-1}) \tag{5.2}$$

where $\mathcal{K}$ is the coefficient often referred to as the coding gain [5]. In the next section, we will express the asymptotic formula of bit error rate in our system, and analyze the performance according to the formula.

5.2 Asymptotic Formula Analysis

From Eq. (4.26), the asymptotic averaged bit error rate $P_e(\mathbf{F})$ can be simplified to an asymptotic formula which can reveal both diversity gain and coding gain as

$$P_e(\mathbf{F}) = \text{CodingGain} \cdot \rho^{-\text{DiversityGain}}$$

where $\mathbf{F}$ is the optimum STBC to be designed and ρ is the SNR.

After complicated calculation (See Appendix F) we derive the asymptotic formula of Eq. (4.26) which has the following form

$$P_e(\mathbf{F}) = C_{-1}(d_1^{-1} + d_2^{-1})\rho^{-1} + O(\rho^{-2}) \tag{5.3}$$

where $C_{-1} = \frac{3}{4}\sqrt{\frac{3}{2\pi}}e^{-\frac{3}{2}} + \frac{43}{8}\sqrt{\frac{3}{2\pi}}E_1\left(\frac{3}{2}\right)$ which is positive, i.e. $C_{-1} > 0$.

From the asymptotic formula in Eq. (F.89) we can see that the diversity gain of our 2×2 MIMO system using zero-forcing equalizer is 1. It has been shown [29] that the diversity gain of a MIMO system equipped with zero-forcing (ZF) receivers and transmitting uncoded symbols is $(N - M + 1)$. Here with our optimum code, the diversity gain is also $N - M + 1$ when $M = N = 2$.

$C_{-1}(d_1^{-1} + d_2^{-1})$ is the coding gain. As $C_{-1} > 0$, to minimize asymptotic error probability $P_e(\mathbf{F})$ is equivalent to minimize $(d_1^{-1} + d_2^{-1})$. From Eq. (F.29), we have $d_1^{-1} + d_2^{-1} = \operatorname{tr}\left(\left(\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H\right)^{-1}\right)$, where d_1, d_2 are the eigenvalues of $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H$. From Eq. (4.14), we have

$$\tilde{\mathbf{F}} = \mathbf{\Sigma}^{\frac{1}{2}}\bar{\mathbf{F}} \tag{5.4}$$

Therefore, $d_1^{-1} + d_2^{-1}$ becomes

$$d_1^{-1} + d_2^{-1} = \mathrm{tr}\left(\left(\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H\right)^{-1}\right)$$

$$= \mathrm{tr}\left(\left(\mathbf{\Sigma}^{\frac{1}{2}}\bar{\mathbf{F}}\bar{\mathbf{F}}^H\mathbf{\Sigma}^{\frac{1}{2}H}\right)^{-1}\right) \tag{5.5}$$

$$= \mathrm{tr}\left(\left(\bar{\mathbf{F}}^{-1}\mathbf{\Sigma}^{-1}\bar{\mathbf{F}}^{-H}\right)^{-1}\right) \tag{5.6}$$

To minimize $d_1^{-1} + d_2^{-1}$ is equivalent to minimize $\mathrm{tr}\left(\left(\bar{\mathbf{F}}^{-1}\mathbf{\Sigma}^{-1}\bar{\mathbf{F}}^{-H}\right)^{-1}\right)$. As for the power constraint,

$$\mathrm{tr}(\mathbf{F}\mathbf{F}^H) = L = 2T$$

$$\Rightarrow \mathrm{tr}(\bar{\mathbf{F}}\bar{\mathbf{F}}^H) = M = 2$$

So combined with Eq. (4.27), the optimization problem becomes

$$\min_{\bar{\mathbf{F}}} \mathrm{tr}\left(\left(\bar{\mathbf{F}}^{-1}\mathbf{\Sigma}^{-1}\bar{\mathbf{F}}^{-H}\right)^{-1}\right)$$

$$\text{s.t. } \mathrm{tr}(\bar{\mathbf{F}}^H\bar{\mathbf{F}}) = 2 \tag{5.7}$$

The optimization problem is exactly the same problem with [1]. This problem has been solved in [1]

$$\bar{\mathbf{F}}_{\mathbf{opt}} = \mathbf{V}_\Sigma \sqrt{\frac{2}{\mathrm{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}}\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}}\mathbf{U}_{\bar{F}} \tag{5.8}$$

where $\mathbf{\Sigma}^{-1} = \mathbf{V}_\Sigma\mathbf{D}_{\Sigma^{-1}}\mathbf{V}_\Sigma^H$ is an eigendecomposition of $\mathbf{\Sigma}^{-1}$, which is the inverse of channel covariance matrix Σ and $\mathbf{U}_{\bar{F}}$ is an arbitrary unitary matrix which will be designed in Chapter 6.

From Eq. (5.4),

$$\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H = \mathbf{\Sigma}^{\frac{1}{2}}\bar{\mathbf{F}}\bar{\mathbf{F}}^H\mathbf{\Sigma}^{\frac{1}{2}H}$$

Applying eigendecomposition to $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H$, $\bar{\mathbf{F}}\bar{\mathbf{F}}^H$ and covariance matrix of channels $\mathbf{\Sigma}$, we have

$$\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H = \mathbf{V}_{\tilde{F}}\mathbf{D}_{\tilde{F}}\mathbf{V}_{\tilde{F}}^H \tag{5.9}$$

$$\bar{\mathbf{F}}\bar{\mathbf{F}}^H = \mathbf{V}_{\bar{F}}\mathbf{D}_{\bar{F}}\mathbf{V}_{\bar{F}}^H \tag{5.10}$$

$$\mathbf{\Sigma} = \mathbf{V}_{\Sigma}\mathbf{D}_{\Sigma}\mathbf{V}_{\Sigma}^H \tag{5.11}$$

where

i $\mathbf{D}_{\tilde{F}}$ is a diagonal matrix with two eigenvalues d_1, d_2;

ii $\mathbf{D}_{\bar{F}}$ is a diagonal matrix with two eigenvalues p_1, p_2;

iii $\mathbf{D}_{\Sigma}$ is a diagonal matrix with two eigenvalues λ_1, λ_2 which are known because $\mathbf{\Sigma}$ is known.

From Eq. (5.8), eigenvectors of $\bar{\mathbf{F}}\bar{\mathbf{F}}^H$ should be equal to eigenvectors of $\mathbf{\Sigma}$ in Eq. (5.9), i.e.

$$\mathbf{V}_{\bar{F}} = \mathbf{V}_{\Sigma} \tag{5.12}$$

Substituting Eq. (5.12) into Eq. (5.9), we obtain

$$\begin{aligned}\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H &= \mathbf{\Sigma}^{\frac{1}{2}}\bar{\mathbf{F}}\bar{\mathbf{F}}^H\mathbf{\Sigma}^{\frac{1}{2}} \\ &= \mathbf{V}_\Sigma\mathbf{D}_\Sigma^{\frac{1}{2}}\mathbf{V}_\Sigma^H\mathbf{V}_{\bar{F}}\mathbf{D}_{\bar{F}}\mathbf{V}_{\bar{F}}^H\mathbf{V}_\Sigma\mathbf{D}_\Sigma^{\frac{1}{2}}\mathbf{V}_\Sigma^H \\ &= \mathbf{V}_\Sigma\mathbf{D}_\Sigma^{\frac{1}{2}}\mathbf{D}_{\bar{F}}\mathbf{D}_\Sigma^{\frac{1}{2}}\mathbf{V}_\Sigma^H \end{aligned} \tag{5.13}$$

Comparing Eq. (5.13) with Eq. (5.9), we obtain

$$\mathbf{D}_{\tilde{F}} = \mathbf{D}_{\bar{F}}\mathbf{D}_\Sigma \tag{5.14}$$

Eigenvalues of $\mathbf{\Sigma}$ are λ_1, λ_2, and eigenvalues of $\bar{\mathbf{F}}\bar{\mathbf{F}}^H$ are p_1, p_2, so we have

$$\begin{cases} d_1 = \lambda_1 p_1 \\ d_2 = \lambda_2 p_2 \end{cases} \tag{5.15}$$

which is an important condition which will be used in Chapter 6.

From Eq. (5.8), we also get the diagonal eigenvalue matrix of $\bar{\mathbf{F}}$ is

$$\begin{pmatrix} \sqrt{p_{1\text{opt}}} & 0 \\ 0 & \sqrt{p_{2\text{opt}}} \end{pmatrix} = \sqrt{\frac{2}{\operatorname{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}}\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}} \tag{5.16}$$

where $\mathbf{D}_{\Sigma^{-1}}$ is the eigenvalue matrix of $\mathbf{\Sigma}$.

In Chapter 6, we will verify that $p_{1\text{opt}}$ and $p_{2\text{opt}}$ in Eq. (5.16) is optimum to minimize BER.

Chapter 6

Evaluation of P_e

6.1 Solution of Optimization Problem

From Eq. (5.16), $p_{1\text{opt}}$ and $p_{2\text{opt}}$, the two eigenvalues of optimum $\bar{\mathbf{F}}\bar{\mathbf{F}}^H$, satisfy

$$\begin{pmatrix} \sqrt{p_{1\text{opt}}} & 0 \\ 0 & \sqrt{p_{2\text{opt}}} \end{pmatrix} = \sqrt{\frac{2}{\operatorname{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}}\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}} \tag{6.1}$$

where $\mathbf{D}_{\Sigma^{-1}}$ is the eigenvalue matrix of $\mathbf{\Sigma}$.

In this chapter, we will verify that $p_{1\text{opt}}$ and $p_{2\text{opt}}$ are correct by evaluating averaged bit error rate P_e in MATLAB.

In Eq. (5.7), there is a connection between eigenvalues of $\bar{\mathbf{F}}\bar{\mathbf{F}}^H$, p_1 and p_2 that $p_1 + p_2 = 2$, so we replace the variables with p by defining that

$$\begin{aligned} p_1 &= p \\ p_2 &= 2 - p \end{aligned} \tag{6.2}$$

where $0 \leq p \leq 2$.

As for $\mathbf{\Sigma}$, which is the covariance matrix of $\mathbf{H}^H\mathbf{H}$, we adopt following exponential correlation model [30] such that the mnth element of $\mathbf{\Sigma}$ is given by:

$$\sigma_{mn} = \begin{cases} \sigma^{n-m}, & m \leq n \\ \sigma^*_{mn}, & m > n \end{cases} \tag{6.3}$$

where σ is the correlation coefficient between any two neighboring antennas and $0 \leq \sigma \leq 1$. $(\cdot)^*$ denotes complex conjugate.

Therefore, in our assumption, when number of transmitter antennas $M = 2$, $\mathbf{\Sigma}$ is a 2×2 matrix which has the form of

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & \sigma \\ \sigma^* & 1 \end{pmatrix} \tag{6.4}$$

when $\sigma = 0$, $\mathbf{H}$ is an uncorrelated channel.

The corresponding eigenvalues of $\mathbf{\Sigma}$ can be calculated (See C.7), so we have

$$\begin{cases} \lambda_1 = 1 + |\sigma| \\ \lambda_2 = 1 - |\sigma| \end{cases} \tag{6.5}$$

If we do the eigendecomposition to the inverse of $\mathbf{\Sigma}$, we obtain

$$\mathbf{\Sigma}^{-1} = \mathbf{V}_{\Sigma^{-1}}\mathbf{D}_{\Sigma^{-1}}\mathbf{V}^H_{\Sigma^{-1}}$$

So that we have the eigenvalue matrix

$$\mathbf{D}_{\Sigma^{-1}} = \begin{pmatrix} \frac{1}{1+|\sigma|} & 0 \\ 0 & \frac{1}{1-|\sigma|} \end{pmatrix} \tag{6.6}$$

Therefore, substituting Eqs. (6.2) and (6.6) into Eq. (6.1), our optimum p_{opt} satisfies that

$$\begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{2-p} \end{pmatrix} = \sqrt{\frac{M}{\operatorname{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}}\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}} \tag{6.7}$$

which means

$$\begin{cases} p = \frac{2}{\sqrt{\frac{1}{1+|\sigma|}}+\sqrt{\frac{1}{1-|\sigma|}}} \times \sqrt{\frac{1}{1+|\sigma|}} \\ 2-p = \frac{2}{\sqrt{\frac{1}{1+|\sigma|}}+\sqrt{\frac{1}{1-|\sigma|}}} \times \sqrt{\frac{1}{1-|\sigma|}} \end{cases} \tag{6.8}$$

where $\operatorname{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right) = \sqrt{\frac{1}{1+|\sigma|}} + \sqrt{\frac{1}{1-|\sigma|}}$.

Next, our aim is to verify the optimum p by evaluating P_e in MATLAB satisfy Eq. (6.8).

Substituting Eqs. (6.2) and (6.5) into Eq. (5.15), the two eigenvalues of $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^H$ can be expressed as

$$\begin{cases} d_1 = \lambda_1 p = (1+|\sigma|)p \\ d_2 = \lambda_2(2-p) = (1-|\sigma|)(2-p) \end{cases}$$

Referring to Eq. (4.25), the objective $P_e(p)$ can be rewritten as

$$\begin{aligned} P_e(p) &= P_e(p_1, p_2) \\ &= \int_0^\infty \frac{u}{\rho^2} \left[\frac{9}{2} e^{-\frac{3}{2}(a-1)} \text{erfc}(\frac{\sqrt{6}}{2}) - \sqrt{\frac{27}{2\pi}} (a^{-2} + a^{-1}) e^{-\frac{3}{2}a} - \frac{3}{2} a^{-\frac{5}{2}} \text{erfc}(\frac{\sqrt{6a}}{2}) \right] du \end{aligned} \tag{6.9}$$

where

$$\begin{aligned} a &= \frac{2(u + d_1^{-1})(u + d_2^{-1})}{\rho u} + 1 \\ &= \frac{2(u + ((1 + |\sigma|)p)^{-1})(u + ((1 - |\sigma|)(2 - p))^{-1})}{\rho u} + 1 \end{aligned}$$

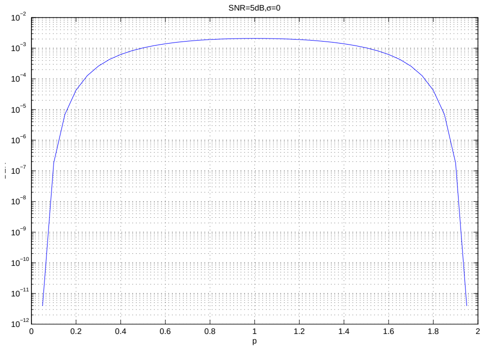
and error function $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$.

In the following, with assistance of numerical simulations in MATLAB, we set the values of signal-to-noise ratio ρ and correlation coefficient σ to find p that minimizes the objective function $P_e(p)$ in each case.

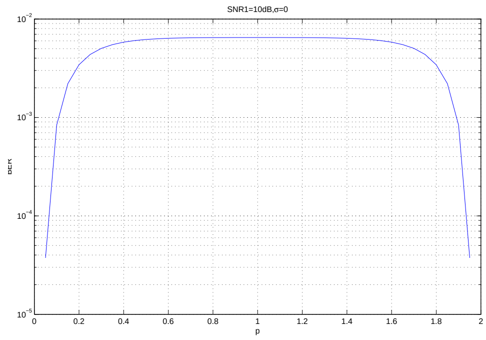
6.1.1 Uncorrelated Channels

We first search for optimum p under different SNRs in uncorrelated channel, i.e., $\sigma = 0$. Under this condition, we sample p within the range from 0 and 2, and then for each SNR from 1dB to 45dB, we calculate values of $P_e(p)$ with different values of p using MATLAB functions such as error function, integration function etc. Therefore, for each SNR, we can find the corresponding p which renders $P_e(p)$ smallest.

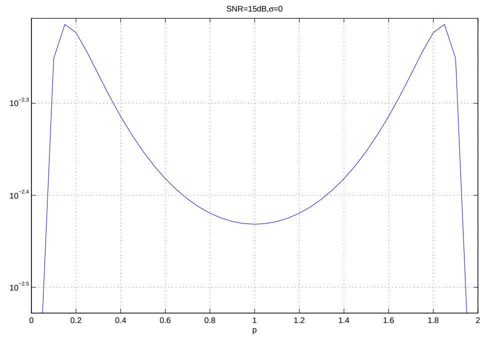
In Figure 6.1, the values of $P_e(p)$ (BER) are shown as p changes, when SNRs are set to different values.



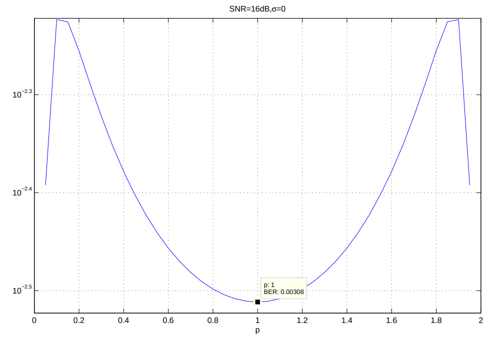
(a) SNR=5dB



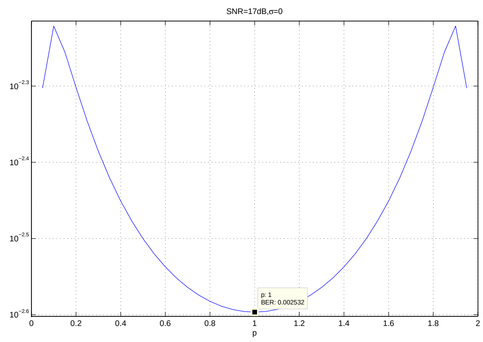
(b) SNR=10dB



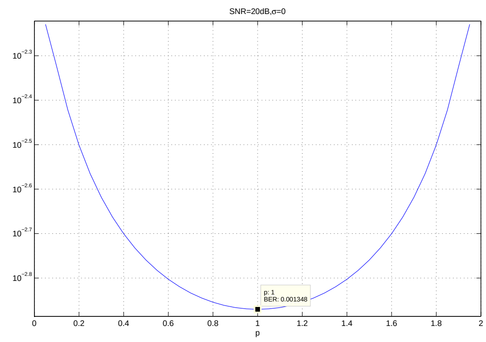
(c) SNR=15dB



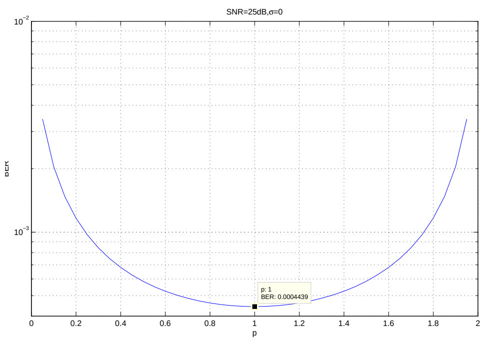
(d) SNR=16dB



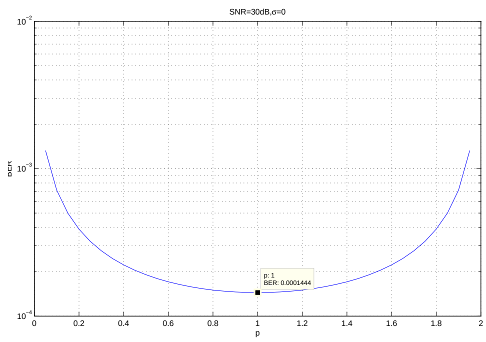
(e) SNR=17dB



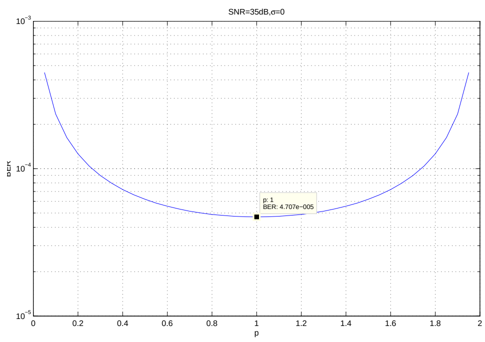
(f) SNR=20dB



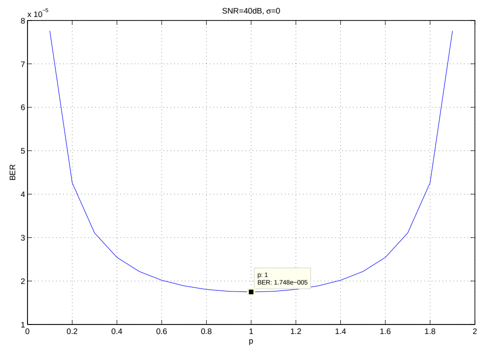
(g) SNR=25dB



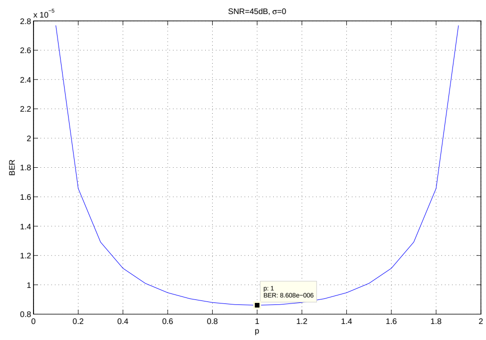
(h) SNR=30dB



(i) SNR=35dB



(j) SNR=40dB



(k) SNR=45dB

Figure 6.1: BER-p plot in uncorrelated channels case ($\sigma = 0$)

From Figure 6.1, we can see that BER is symmetric with the symmetric axis $p = 1$. And when SNR $\rho < 16$dB, $p_{\text{opt}} = 0$ or $p_{\text{opt}} = 2$, $P_e(p)$ (BER) has the smallest value; on the other hand, when SNR $\rho \geq 16$dB, $p_{\text{opt}} = 1$, $P_e(p)$ (BER) has the smallest value where p_{opt} means the optimum p.

So in the uncorrelated channel case, we can draw a plot of p_{opt} to SNR as Figure 6.2

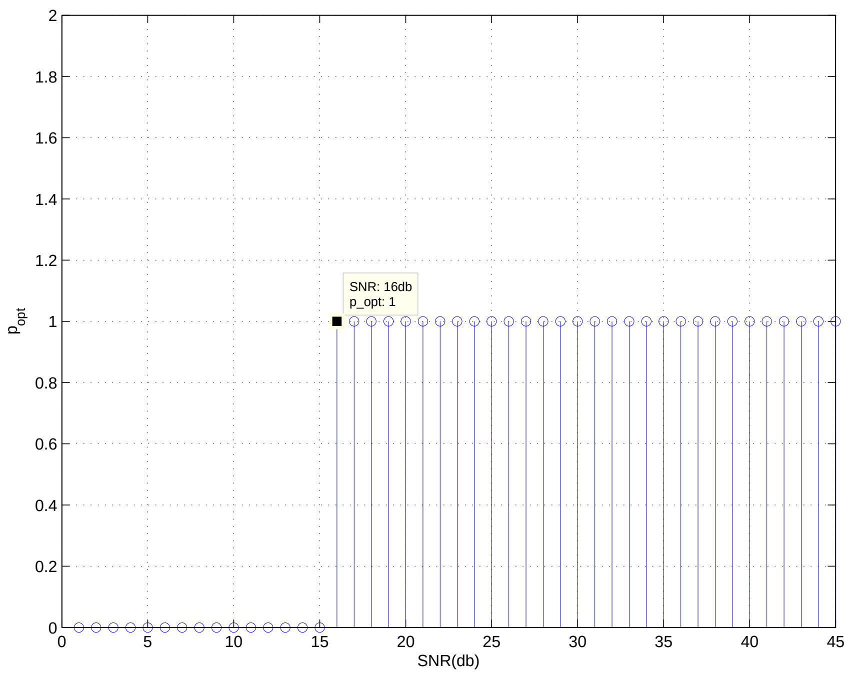


Figure 6.2: p_{opt}-SNR plot in uncorrelated channels case ($\sigma = 0$)

If we draw a plot to see the relationship between SNR and BER when $\rho \geq 16$dB and $p_{\text{opt}} = 1$, we can see that as SNR goes up, BER decreases accordingly (Figure 6.3).

Therefore, in the case of uncorrelated channels, the solution is $p_{\text{opt}} = 1$ with condition $\rho \geq 16$dB to satisfy the conditions of convex functions as follows:

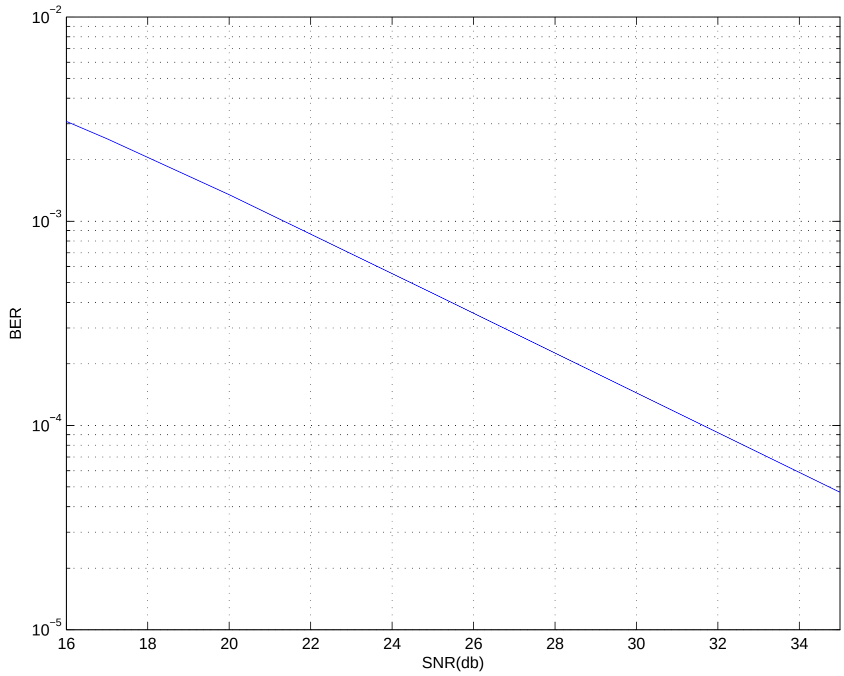


Figure 6.3: BER-SNR plot in uncorrelated channels case ($\sigma = 0$)

$$0 < [\mathbf{V}_{be}]_{ii} = [\mathbf{V}_{be}]_{jj} \leq \frac{1}{3} \; \forall i,\, j = 1,\, \ldots,\, 2L. \tag{6.10}$$

6.1.2 Moderate Channel Correlation

Secondly, we search for optimum p under different SNRs in moderately correlated channel, e.g., $|\sigma| = 0.2$ and $|\sigma| = 0.5$. Similarly, we sample p within the range from 0 and 2, and then for each SNR from 1dB to 45dB, we calculate values of $P_e(p)$ with different values of p using MATLAB functions such as error function, integration function etc. Therefore, for each SNR, we can find the corresponding p which renders $P_e(p)$ smallest.

a) Case $|\sigma| = 0.2$:

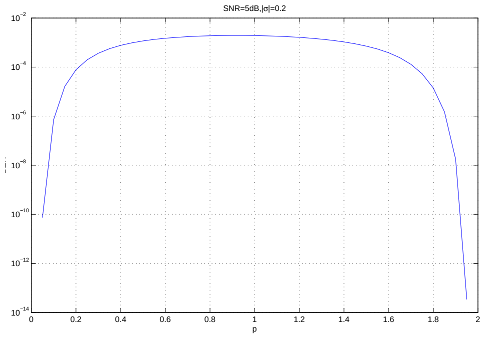
In Figure 6.4, the values of $P_e(p)$ (BER) are shown as p changes, when SNRs are set to different values.

From Figure 6.4, we can see that when SNR $\rho < 17$dB, $p_{\text{opt}} = 2$, $P_e(p)$ (BER) has the smallest value; on the other hand, when SNR $\rho \geq 17$dB, $p_{\text{opt}} = 0.9$, $P_e(p)$ (BER) has the smallest value where p_{opt} means the optimum p.

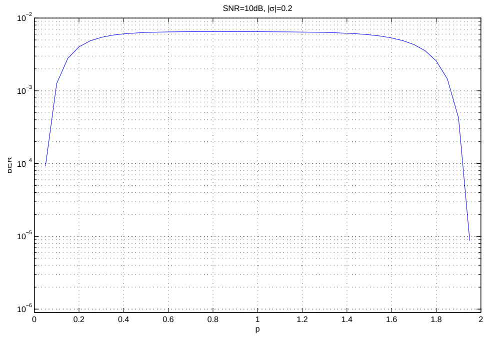
So in the moderately correlated channel $|\sigma| = 0.2$ case, we can draw a plot of p_{opt} to SNR as Figure 6.5.

If we draw a plot to see the relationship between SNR and BER when $\rho \geq 17$dB and $p_{\text{opt}} = 0.9$, we can see that as SNR goes up, BER decreases accordingly (Figure 6.6).

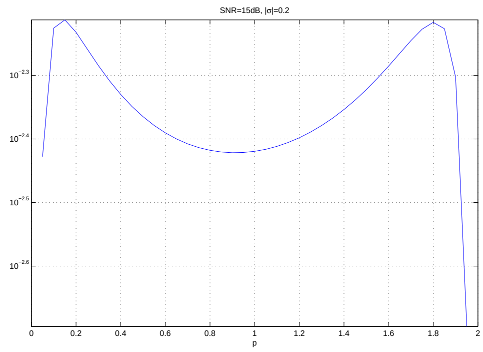
Therefore, in the case of moderately correlated channel where $|\sigma| = 0.2$, the solution is $p_{\text{opt}} = 0.9$ with condition $\rho \geq 17$dB to satisfy the conditions of convex



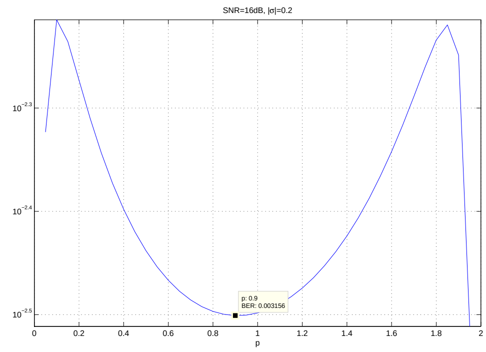
(a) SNR=5dB



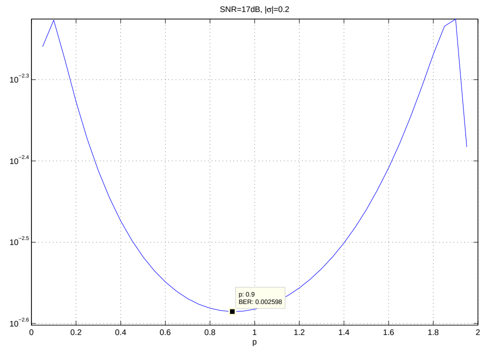
(b) SNR=10dB



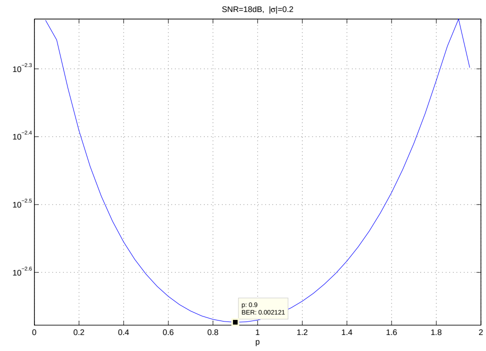
(c) SNR=15dB



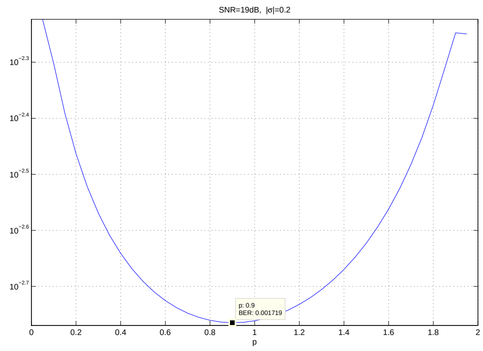
(d) SNR=16dB



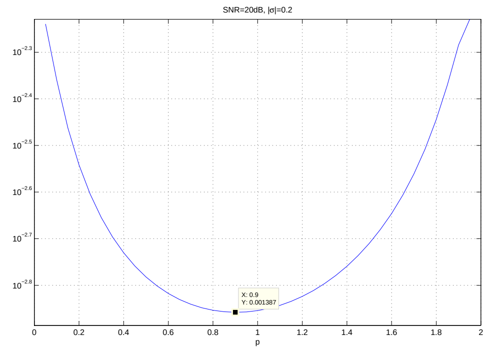
(e) SNR=17dB



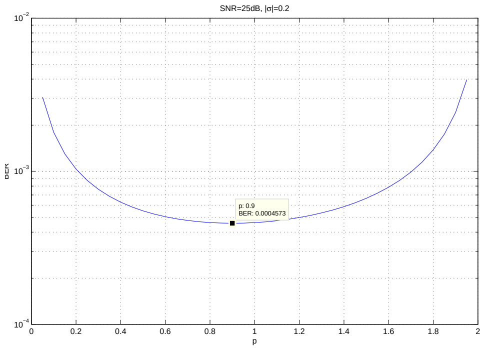
(f) SNR=18dB



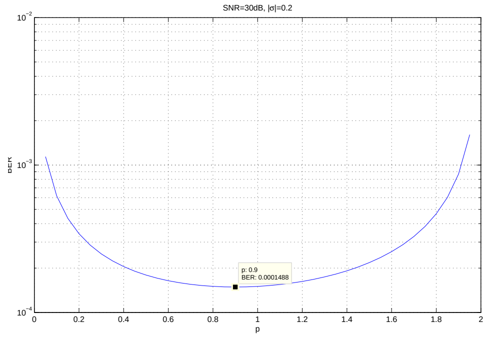
(g) SNR=19dB



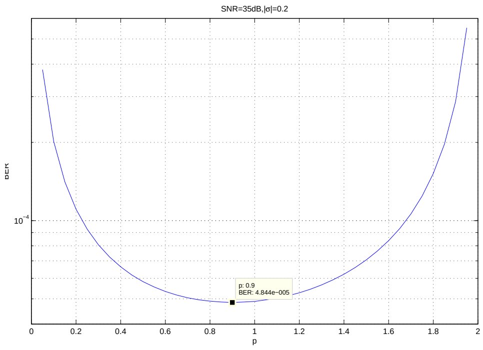
(h) SNR=20dB



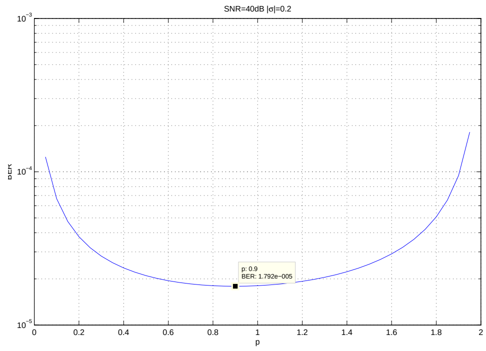
(i) SNR=25dB



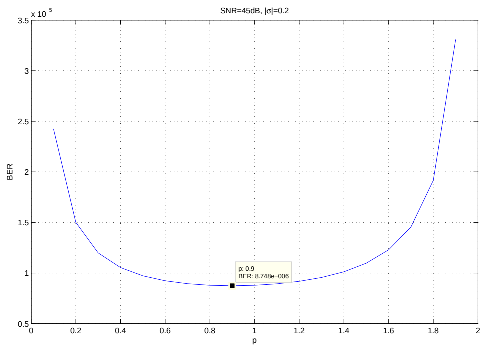
(j) SNR=30dB



(k) SNR=35dB



(l) SNR=40dB



(m) SNR=45dB

Figure 6.4: BER-p plot in moderately correlated channels case ($|\sigma| = 0.2$)

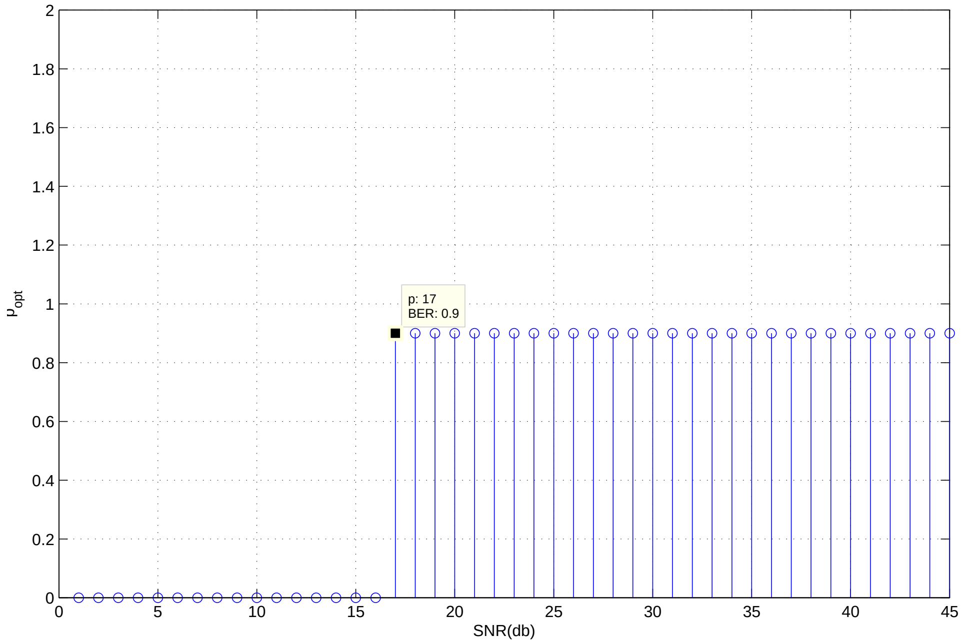


Figure 6.5: p_{opt}-SNR plot in moderately correlated channels case ($|\sigma| = 0.2$)

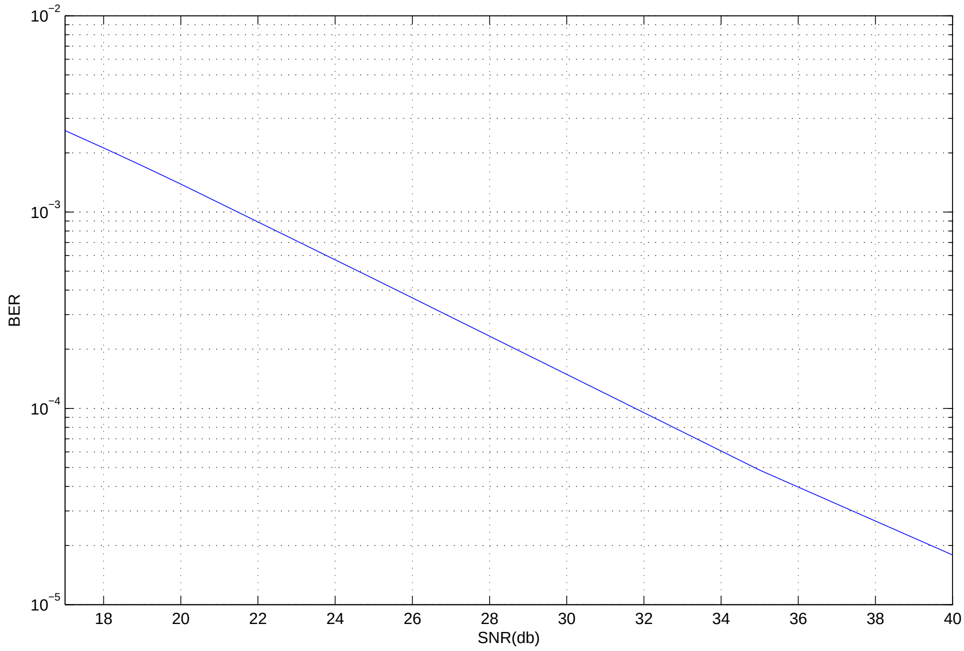


Figure 6.6: BER-SNR plot in moderately correlated channels case ($|\sigma| = 0.2$)

functions.

b) Case $|\sigma| = 0.5$:

In Figure 6.7, the values of $P_e(p)$ (BER) are shown as p changes, when SNRs are set to different values.

From Figure 6.7, we can see that when SNR $\rho < 18$dB, $p_{\text{opt}} = 2$, $P_e(p)$ (BER) has the smallest value; on the other hand, when SNR $\rho \geq 18$dB, $p_{\text{opt}} = 0.75$, $P_e(p)$ (BER) has the smallest value where p_{opt} means the optimum p.

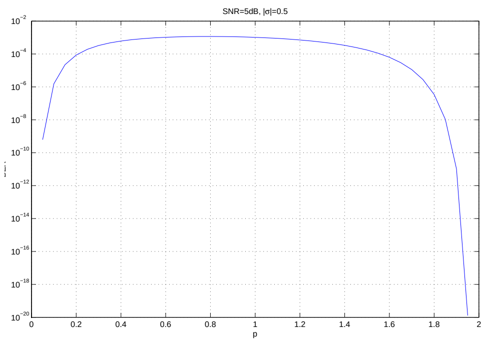
So in the moderately correlated channel $|\sigma| = 0.5$ case, we can draw a plot of p_{opt} to SNR as Figure 6.8

If we draw a plot to see the relationship between SNR and BER when $\rho \geq 18$dB and $p_{\text{opt}} = 0.75$, we can see that as SNR goes up, BER decreases accordingly (Figure 6.9).

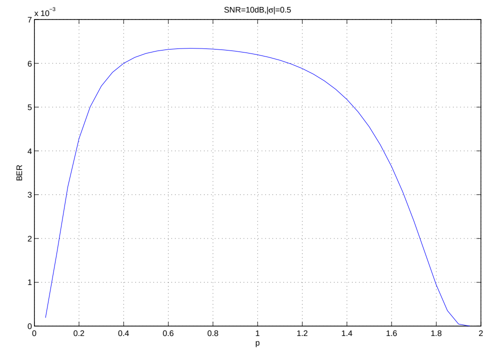
Therefore, in the case of moderately correlated channel where $|\sigma| = 0.5$, the solution is $p_{\text{opt}} = 0.75$ with condition $\rho \geq 18$dB to satisfy the conditions of convex functions

6.1.3 Higher Channel Correlation

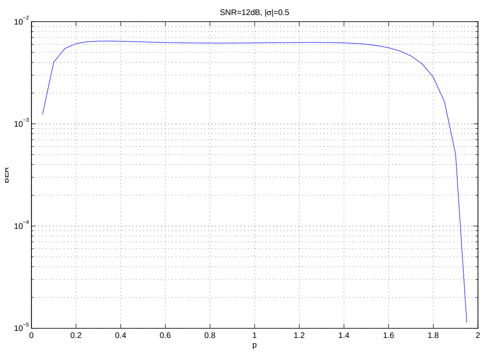
Lastly, we search for optimum p under different SNRs in highly correlated channel, e.g., $|\sigma| = 0.8$. Similarly, we sample p within the range from 0 and 2, and then for each SNR from 1dB to 45dB, we calculate values of $P_e(p)$ with different ps using MATLAB functions such as error function, integration function etc. Therefore, for each SNR, we can find the corresponding p which renders $P_e(p)$ smallest. In Figure 6.10, the



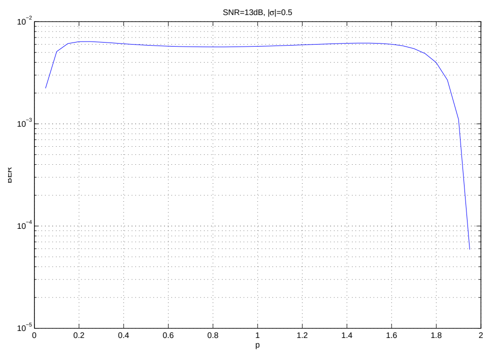
(a) SNR=5dB



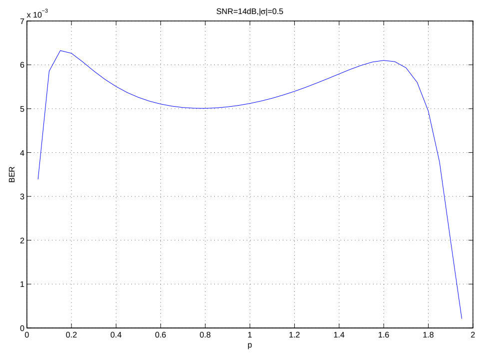
(b) SNR=10dB



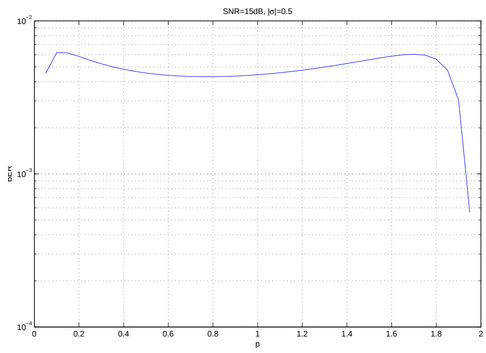
(c) SNR=12dB



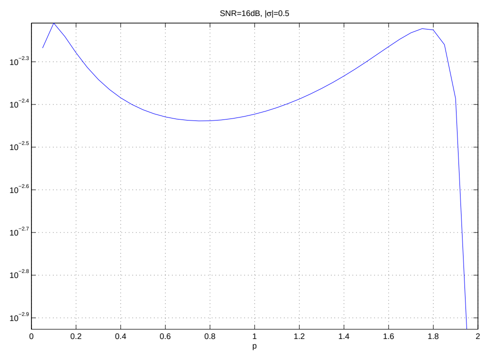
(d) SNR=13dB



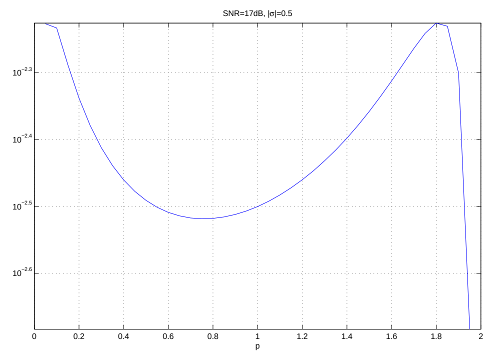
(e) SNR=14dB



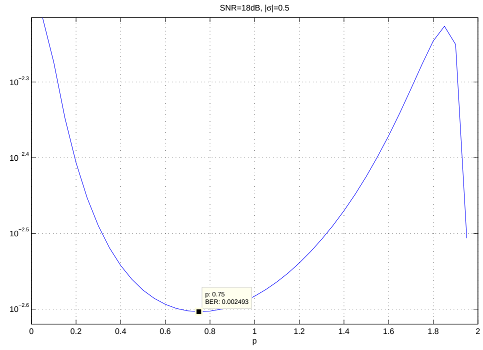
(f) SNR=15dB



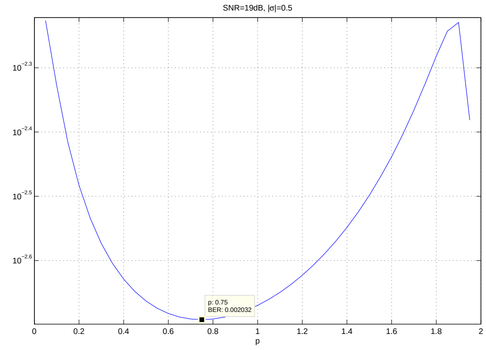
(g) SNR=16dB



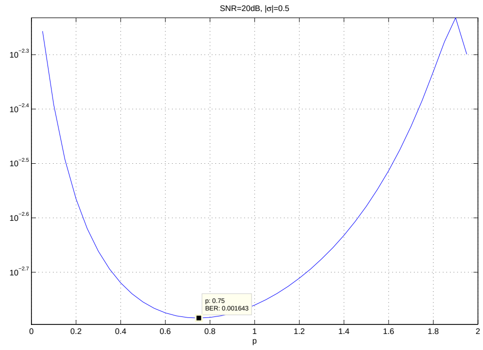
(h) SNR=17dB



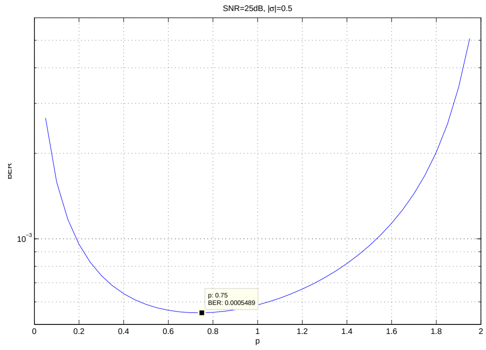
(i) SNR=18dB



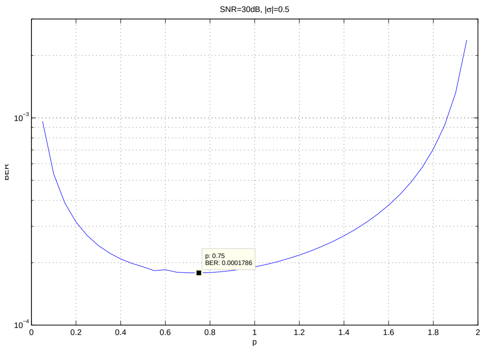
(j) SNR=19dB



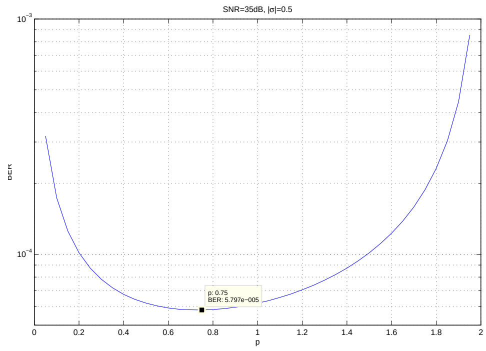
(k) SNR=20dB



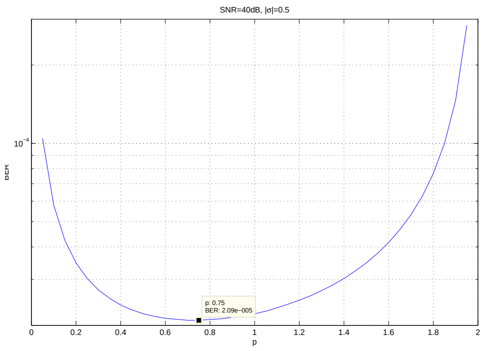
(l) SNR=25dB



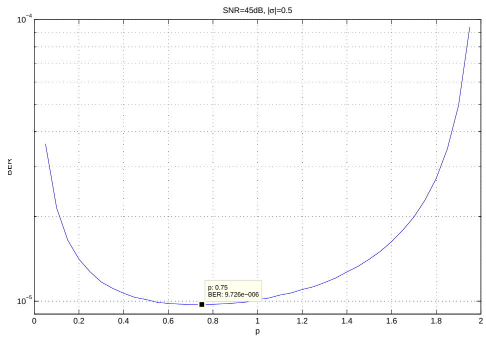
(m) SNR=30dB



(n) SNR=35dB



(o) SNR=40dB



(p) SNR=45dB

Figure 6.7: BER-p plot in moderately correlated channels case ($|\sigma| = 0.5$)

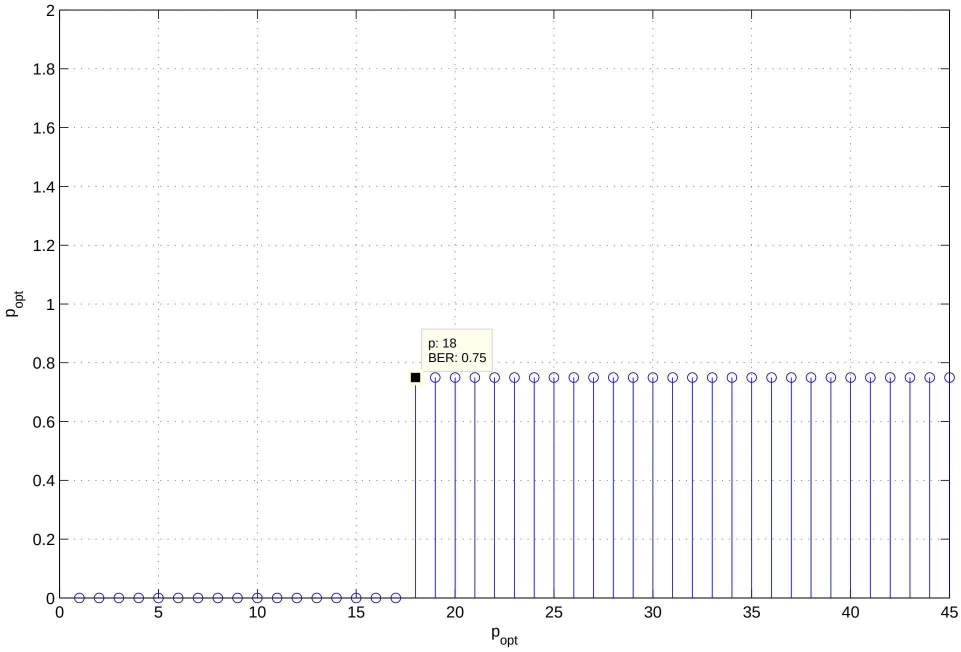


Figure 6.8: p_{opt}-SNR plot in moderately correlated channels case ($|\sigma| = 0.5$)

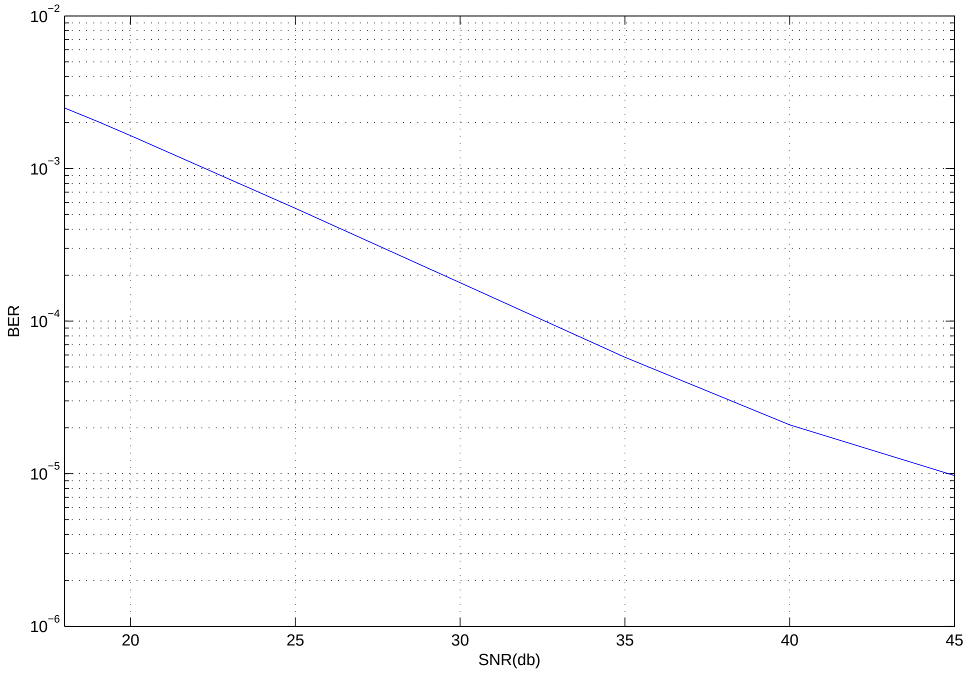


Figure 6.9: BER-SNR plot in moderately correlated channels case ($|\sigma| = 0.5$)

values of $P_e(p)$ (BER) are shown as p changes, when SNRs are set to different values.

From Figure 6.10, we can see that when SNR $\rho < 22$dB, $p_{\text{opt}} = 2$, $P_e(p)$ (BER) has the smallest value; on the other hand, when SNR $\rho \geq 22$dB, $p_{\text{opt}} = 0.5$, $P_e(p)$ (BER) has the smallest value where p_{opt} means the optimum p.

So in the highly correlated channel $|\sigma| = 0.8$ case, we can draw a plot of p_{opt} to SNR as Figure 6.11

If we draw a plot to see the relationship between SNR and BER when $\rho \geq 22$dB and $p_{\text{opt}} = 0.5$, we can see that as SNR goes up, BER decreases accordingly (Figure 6.12).

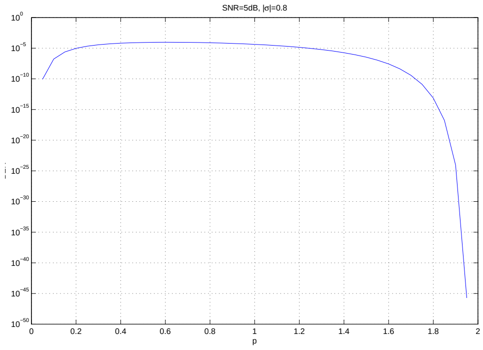
Therefore, in the case of highly correlated channel where $|\sigma| = 0.8$, the solution is $p_{\text{opt}} = 0.5$ with condition $\rho \geq 22$dB to satisfy the conditions of convex functions.

When we select the optimum p_{opt}s for the four channel cases, relationship between SNR and BER can be shown together in Figure 6.13. It is shown that the channel is more correlated, BER is higher.

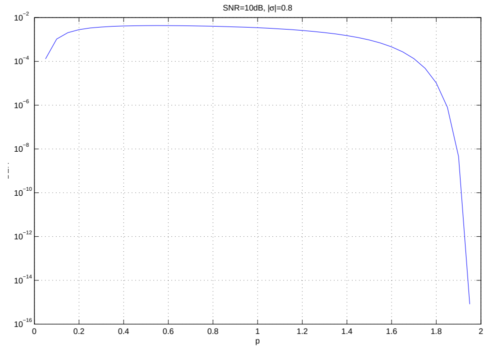
Above all, from the four cases, we can get the pair of absolute value of channel correlation coefficient $|\sigma|$ and optimum p_{opt} as follows:

$\|\sigma\|$	p_{opt}
0	1
0.2	0.9
0.5	0.75
0.8	0.5

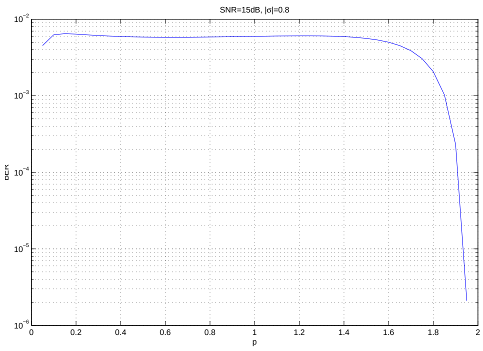
Substituting these values of p_{opt} and $|\sigma|$ into Eq. (6.8), we find that equation is



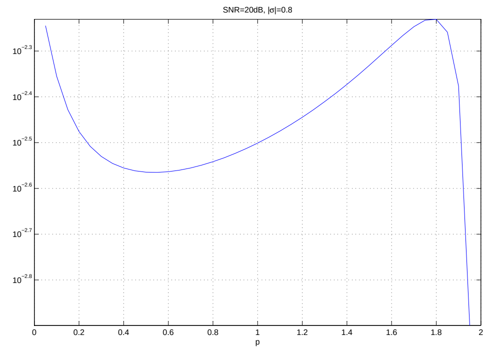
(a) SNR=5dB



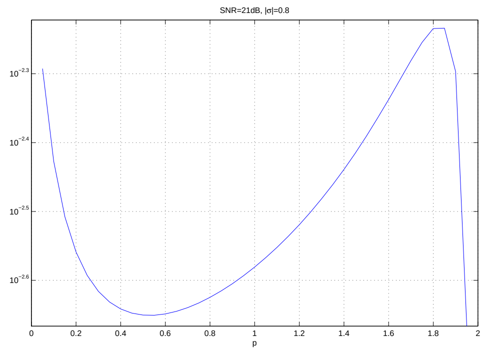
(b) SNR=10dB



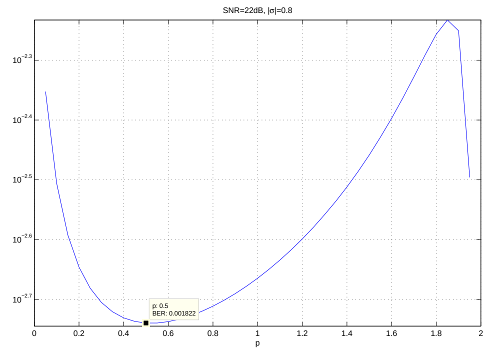
(c) SNR=15dB



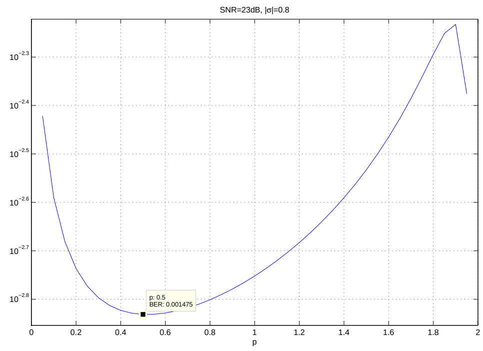
(d) SNR=20dB



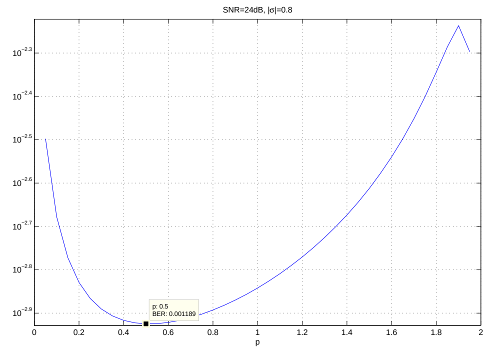
(e) SNR=21dB



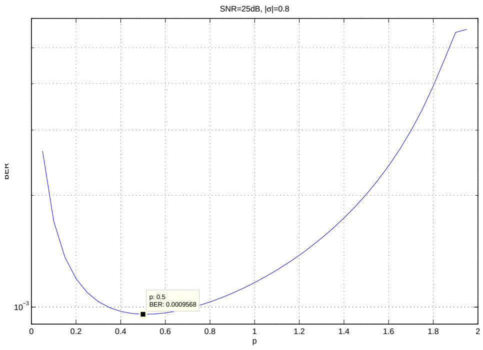
(f) SNR=22dB



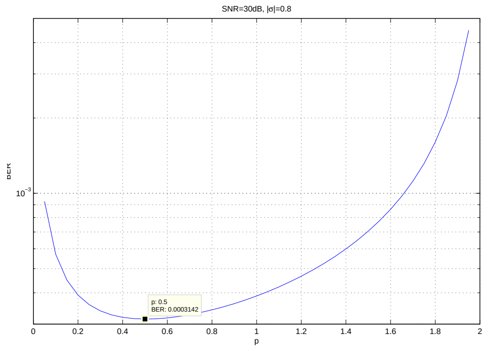
(g) SNR=23dB



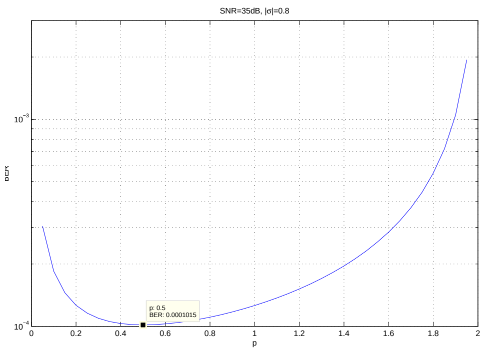
(h) SNR=24dB



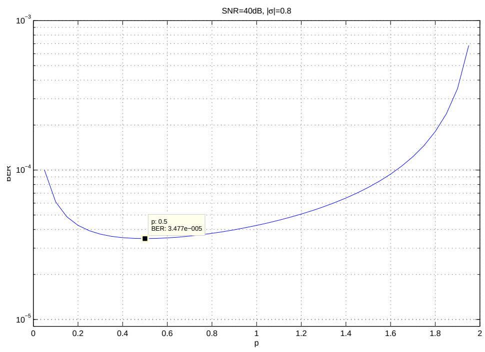
(i) SNR=25db



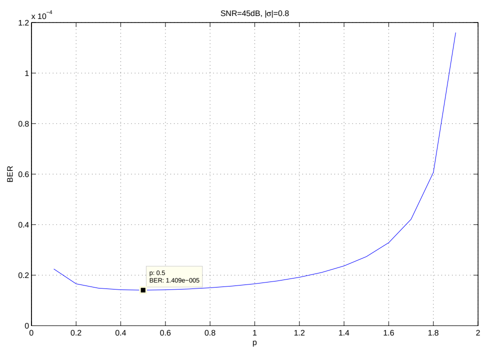
(j) SNR=30db



(k) SNR=35db



(l) SNR=40db



(m) SNR=45db

Figure 6.10: BER-p plot in moderately correlated channels case ($|\sigma| = 0.8$)

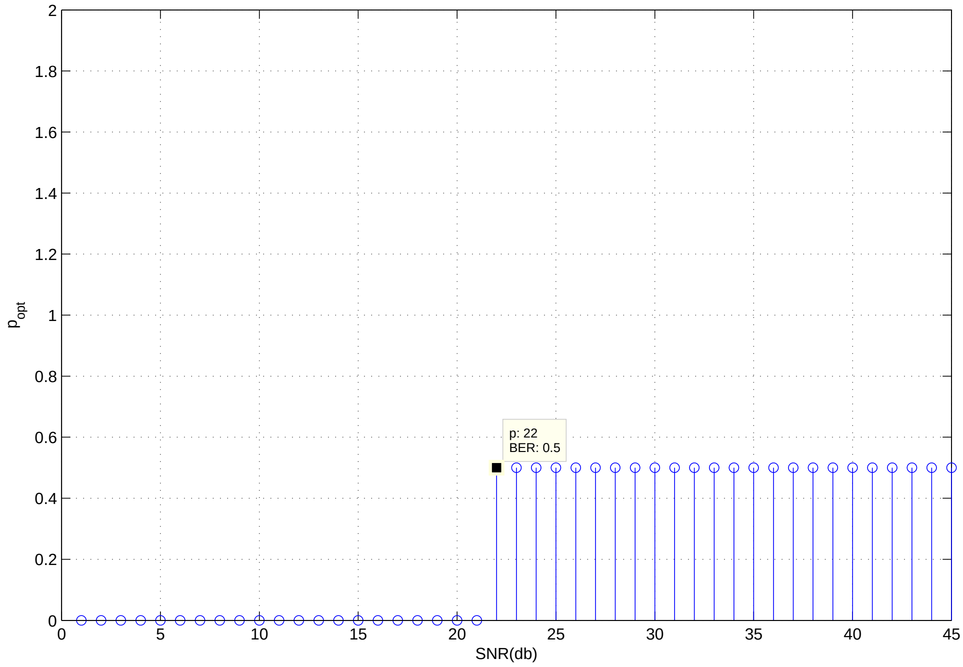


Figure 6.11: p_{opt}-SNR plot in highly correlated channels case ($|\sigma| = 0.8$)

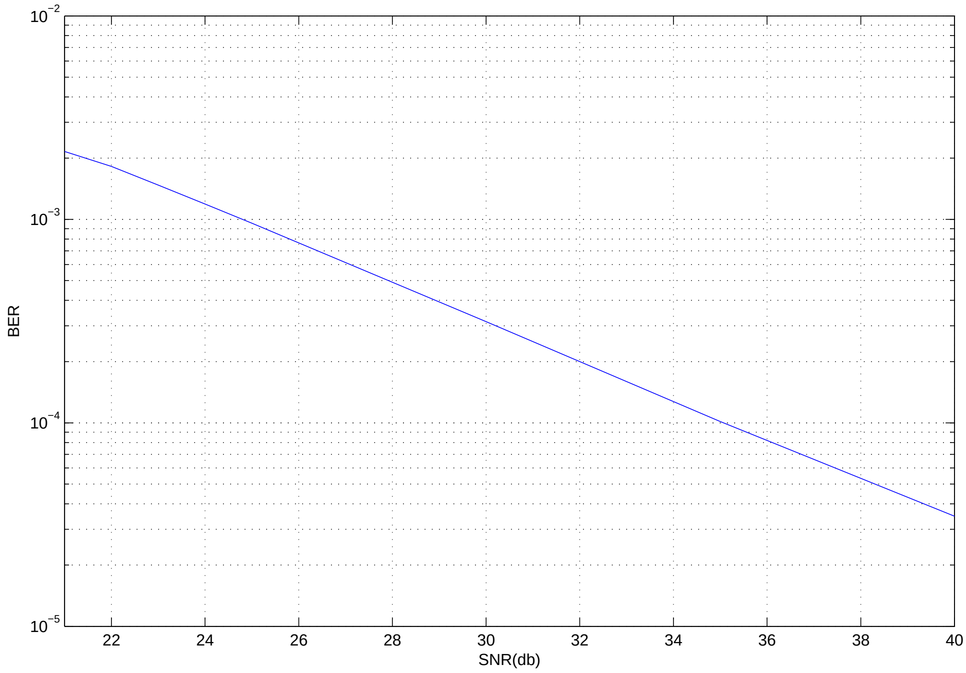


Figure 6.12: BER-SNR plot in highly correlated channels case ($|\sigma| = 0.8$)

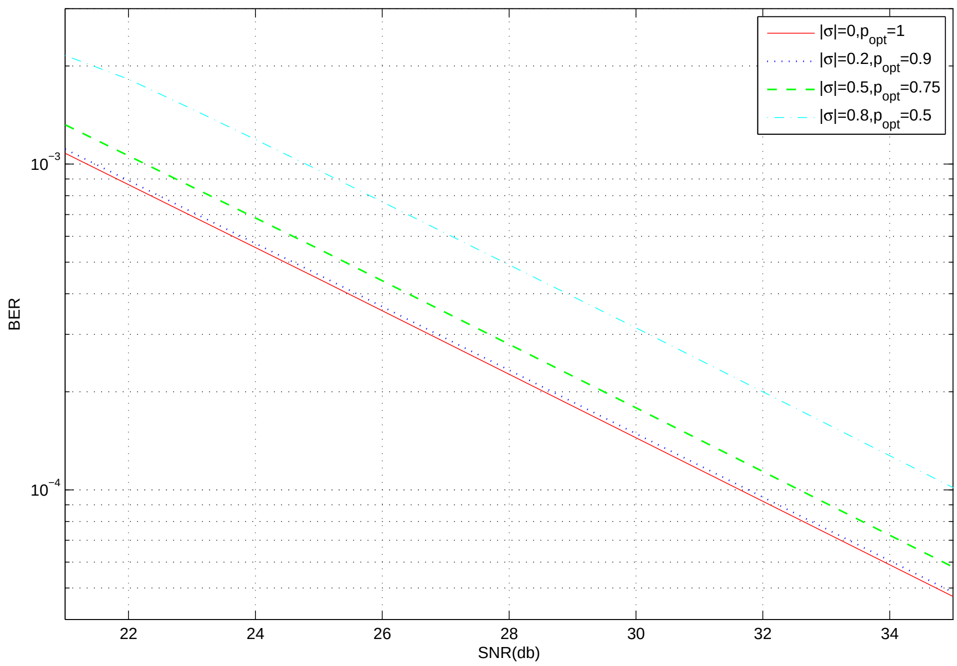


Figure 6.13: BER-SNR plot in different channel cases

exactly satisfied, which verifies the correctness of choosing p_{opt}.

Now we have solved the simplified optimization problem to choose $\bar{\mathbf{F}}$ as in Eq. (5.7), in next section we will combine all the conditions to achieve the minimum value of asymptotic bit error rate to find the solution of original problem in Eq. (3.9), i.e. to find $\mathbf{F}$.

6.2 Structure of Optimum Code

The structure of optimum code F should satisfy:

1. From Eq. (4.3) and condition of convex function, we have

$$0 < [\mathbf{V}_{be}]_{ii} = [\mathbf{V}_{be}]_{jj} \leq \frac{1}{3}, \forall\, i,\, j = 1,\ \ldots,\ 2L \tag{6.11}$$

$$\begin{aligned}\mathbf{V}_{be} &= \mathrm{E}[(\sigma - \hat{\sigma})(\sigma - \hat{\sigma})^H] \\ &= \frac{M}{\rho}[\mathbf{T}^H\hat{\mathbf{F}}^H\hat{\mathbf{H}}^H\hat{\mathbf{H}}\hat{\mathbf{F}}\mathbf{T}]^{-1}\end{aligned}$$

where $2L \times 2L$ matrices $\hat{\mathbf{H}}$, $\hat{\mathbf{F}}$, $\mathbf{T}$ are defined as

$$\hat{\mathbf{H}} \triangleq \begin{pmatrix} \mathbf{I}_T \otimes \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_T \otimes \mathbf{H}^* \end{pmatrix} \tag{6.12}$$

$$\hat{\mathbf{F}} \triangleq \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^* \end{pmatrix} \tag{6.13}$$

$$\mathbf{T} \triangleq \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I}_L & j\mathbf{I}_L \\ \mathbf{I}_L & -j\mathbf{I}_L \end{pmatrix} \tag{6.14}$$

$[\mathbf{V}_{be}]_{ll}$, $l = 1, \ldots, 2L$, is the MSE of the lth bit of σ, which means the equalized MSEs for each symbol must be all equal.

2. From Eq. (4.9), the optimum code we design has the structure

$$\mathbf{F} = \begin{pmatrix} \bar{\mathbf{F}} & 0 & \cdots & 0 \\ 0 & \bar{\mathbf{F}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \bar{\mathbf{F}} \end{pmatrix} \mathbf{V}_F^H \tag{6.15}$$

where $\mathbf{V}_F^H$ is an arbitrary unitary matrix.

In the following conditions, we will decide $\bar{\mathbf{F}}$ and $\mathbf{V}_F^H$ respectively.

3. As for $\bar{\mathbf{F}}$, in Eq. (5.8), we have

$$\bar{\mathbf{F}}_{\mathbf{opt}} = \mathbf{V}_\Sigma \sqrt{\frac{2}{\operatorname{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}} \mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}} \mathbf{U}_{\bar{F}} \tag{6.16}$$

where $\mathbf{\Sigma}^{-1} = \mathbf{V}_\Sigma \mathbf{D}_{\Sigma^{-1}} \mathbf{V}_\Sigma^H$ is an eigendecomposition of $\mathbf{\Sigma}^{-1}$, which is the inverse of channel covariance matrix Σ and $\mathbf{U}_{\bar{F}}$ is an arbitrary unitary matrix.

From Eq. (8.1), we can see there is still a $\mathbf{V}_F^H$ behind block diagonal matrix, so we can define $\mathbf{U}_{\bar{F}}$ as an identity matrix. Therefore,

$$\bar{\mathbf{F}} = \mathbf{V}_{\Sigma}\sqrt{\frac{2}{\operatorname{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}}\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}}\mathbf{I} \tag{6.17}$$

4. Now we will find $\mathbf{V}_F^H$ in Eq. (8.1).

$$\begin{aligned}\operatorname{tr}(\mathbf{F}^H\mathbf{F}) &= \operatorname{tr}\left[\mathbf{V}_F\begin{pmatrix}\bar{\mathbf{F}} & 0 & \cdots & 0\\ 0 & \bar{\mathbf{F}} & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & \bar{\mathbf{F}}\end{pmatrix}^H\begin{pmatrix}\bar{\mathbf{F}} & 0 & \cdots & 0\\ 0 & \bar{\mathbf{F}} & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & \bar{\mathbf{F}}\end{pmatrix}\mathbf{V}_F^H\right]\\ &= \operatorname{tr}\left[\begin{pmatrix}\bar{\mathbf{F}}^H\bar{\mathbf{F}} & 0 & \cdots & 0\\ 0 & \bar{\mathbf{F}}^H\bar{\mathbf{F}} & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & \bar{\mathbf{F}}^H\bar{\mathbf{F}}\end{pmatrix}\right]\end{aligned} \tag{6.18}$$

From Eq. (6.17) we can derive that

$$\begin{aligned}\bar{\mathbf{F}}^H\bar{\mathbf{F}} &= \sqrt{\frac{2}{\operatorname{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}}\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}H}\mathbf{V}_{\Sigma}^H\mathbf{V}_{\Sigma}\sqrt{\frac{2}{\operatorname{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}}\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}}\\ &= \frac{2}{\operatorname{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\end{aligned} \tag{6.19}$$

which is a diagonal matrix. Therefore, in Eq. (6.18), the matrix inside trace is also diagonal.

From Chapter 2, $\mathbf{F} = [\operatorname{vec}(\mathbf{C}_1)\cdots\operatorname{vec}(\mathbf{C}_L)] : MT \times MT$, where $\mathbf{C}_1$, …, $\mathbf{C}_L$ are

$M \times T$ coding matrices, so we have

$$\mathrm{tr}(\mathbf{F}^H\mathbf{F}) = \mathrm{tr}\begin{pmatrix} \mathrm{vec}(\mathbf{C}_1)^H \\ \vdots \\ \mathrm{vec}(\mathbf{C}_L)^H \end{pmatrix}\begin{pmatrix} \mathrm{vec}(\mathbf{C}_1) & \cdots & \mathrm{vec}(\mathbf{C}_L) \end{pmatrix}$$
$$= \mathrm{tr}\begin{pmatrix} |\mathrm{vec}(\mathbf{C}_1)|^2 & 0 & \cdots & 0 \\ 0 & |\mathrm{vec}(\mathbf{C}_2)|^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & |\mathrm{vec}(\mathbf{C}_L)|^2 \end{pmatrix}$$

where $\mathrm{tr}[\mathbf{C}_i^H\mathbf{C}_l] = \mathrm{tr}[\mathrm{vec}(\mathbf{C}_i)^H\mathrm{vec}(\mathbf{C})_l] = 0$ when $i \neq l$. $\mathbf{C}_l$ is defined as trace-orthogonal to each other.

Therefore, $\mathbf{V}_F^H$ is a unitary matrix which can make $\mathbf{C}_l$ trace-orthogonal. [15] presents an algorithm for the generation of codes possesing the trace-orthogonal properties, which will be shown in Chapter 7.

Above all, we have got the optimum code $\mathbf{F}$. In Chapter 7, we will build a MIMO system model and generate the optimum STBC to perform some simulations in MATLAB.

Chapter 7

Simulation Results

7.1 Simulation Model

In the following examples, simulations were carried out for a MIMO system with 2 transmitter antennas and 2 receiver antennas transmitting symbols from a 4-QAM constellation. For each randomly generated channel matrix $\mathbf{H}$, the experiment was carried out for different SNR, and for each SNR, was repeated 10^8 times with different noise realizations. The average bit error rate (BER) was then computed for the various values of SNR. To study the effect of antenna correlations, random realizations of correlated channels were generated with various values of correlation coefficient σ such that $\sigma = 0$, $0.2e^{0.5j}$, $0.5e^{0.5j}$, $0.8e^{0.5j}$, $0.999e^{0.5j}$ where σ is complex.

7.2 Optimum Code Generation

From Eq. (5.10), our optimum code has following structure:

$$\mathbf{F} = \begin{pmatrix} \bar{\mathbf{F}} & 0 & \cdots & 0 \\ 0 & \bar{\mathbf{F}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \bar{\mathbf{F}} \end{pmatrix} \mathbf{V}_F^H \tag{7.1}$$

where $\bar{\mathbf{F}} = \mathbf{V}_\Sigma \sqrt{\frac{2}{\mathrm{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}} \mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}} \mathbf{I}$ and $\mathbf{V}_F^H = [\mathrm{vec}(\mathbf{U}_1)\mathrm{vec}(\mathbf{U}_2)\cdots\mathrm{vec}(\mathbf{U}_L)]$ is a unitary matrix using following steps.

- First realign $\mathbf{U}_l$, $l = 1, ..., L$ to $\mathbf{U}_{mt}$, $m = 1, ..., M$ and $t = 1, ..., T$.
- Form a $T \times T$ row permutation matrix such that

$$\mathbf{P} = \begin{bmatrix} \mathbf{0} & 1 \\ \mathbf{I}_{T-1} & \mathbf{0} \end{bmatrix} \tag{7.2}$$

- Form a $M \times M$ DFT matrix $D = [\mathbf{d}_1 \mathbf{d}_2 \cdots \mathbf{d}_M]$ with $\mathbf{d}_m$ being mth column.
- Final coding matrices

$$\mathbf{U}_{mt} = [\mathrm{diag}(\mathbf{d}_m)|\mathbf{0}]\mathbf{P}^{t-1} \qquad m = 1, ..., M; t = 1, ..., T \tag{7.3}$$

7.3 Simulation Results

Applying the generated optimum codes on our MIMO system model, the bit error rate corresponding to signal-to-noise ratio in different channel scenarios has been shown in Figure 7.1. We can see that BER decreases as SNR is increasing. When channel is more correlated, BER is higher with the same SNR. The simulation result is the same with our numerical evaluations in Figure 6.13.

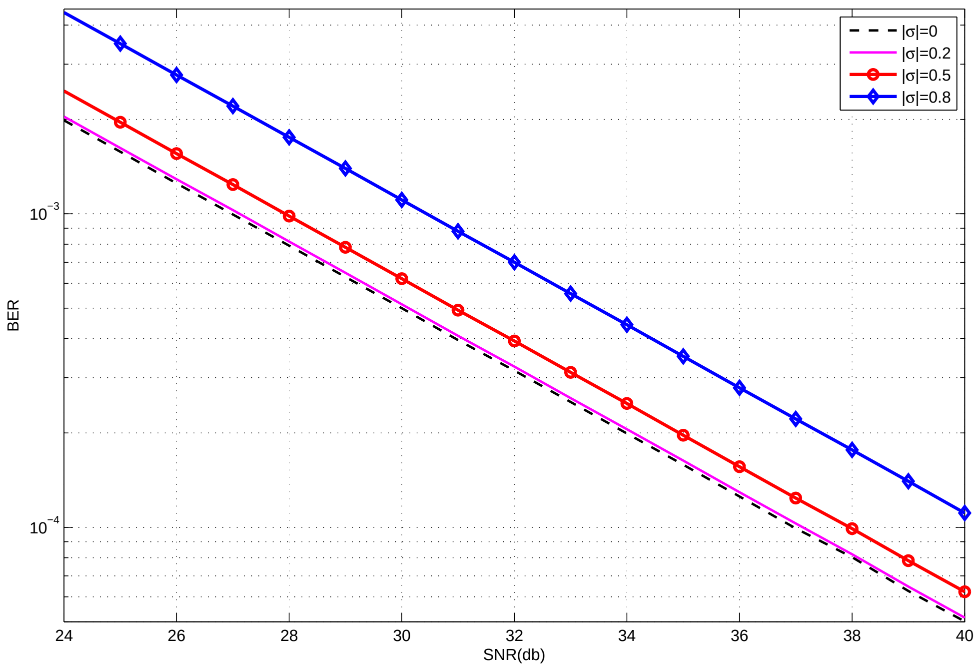


Figure 7.1: SNR-BER when $\sigma = 0$, $0.2e^{0.5j}$, $0.5e^{0.5j}$, $0.8e^{0.5j}$

7.4 Comparison of Optimum Code and Uncoded System

If we put the optimum code **F** into identity matrix, compare the BER of uncoded system with our optimum code system in uncorrelated channel, moderately correlated channel and highly correlated channels. Figure 7.2 and Figure 7.3 show the uncorrelated channel scenario and moderately correlated channel scenario respectively. Figures 7.4, 7.5 and 7.6 shows that in higher correlated channels, system with optimum code has better performance than uncoded system. We can also see that as channel is more correlated, the advantage of our optimum code is more apparent.

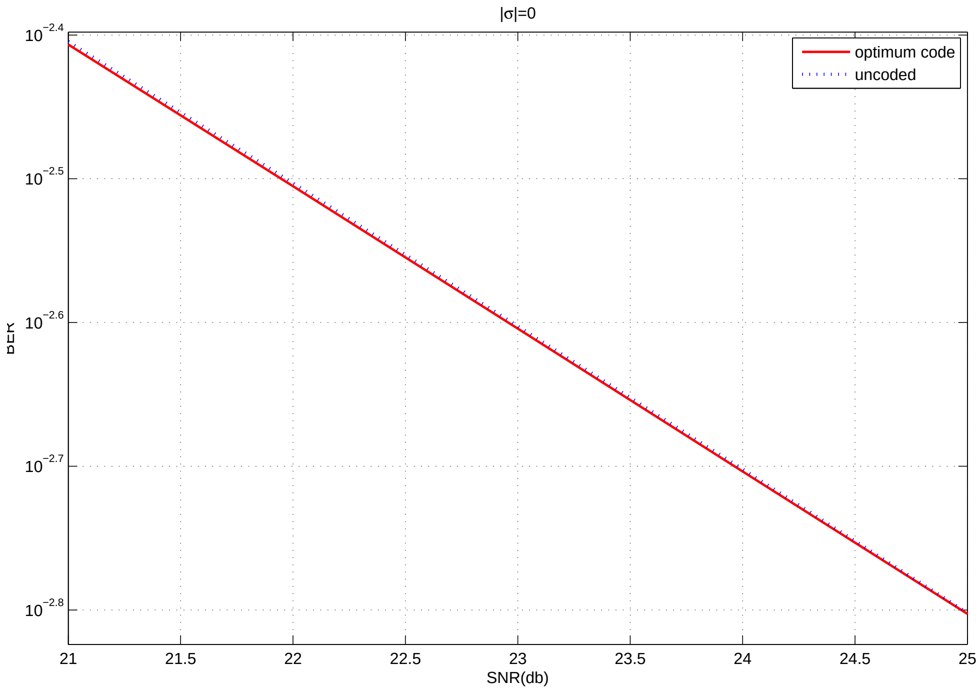


Figure 7.2: Comparison of SNR-BER when $\sigma = 0$

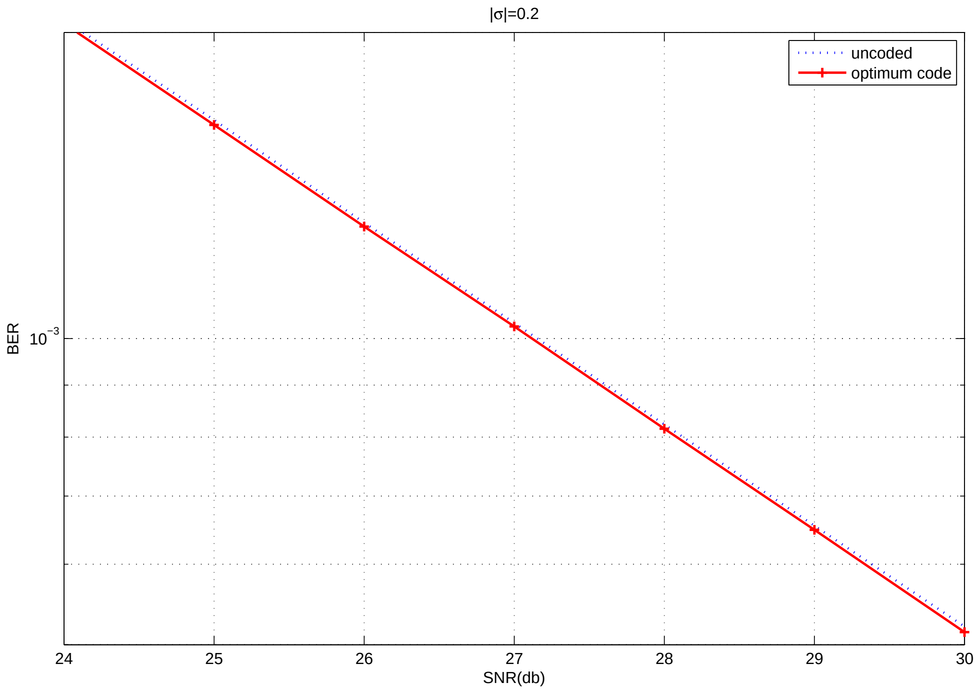


Figure 7.3: Comparison of SNR-BER when $\sigma = 0.2e^{0.5j}$

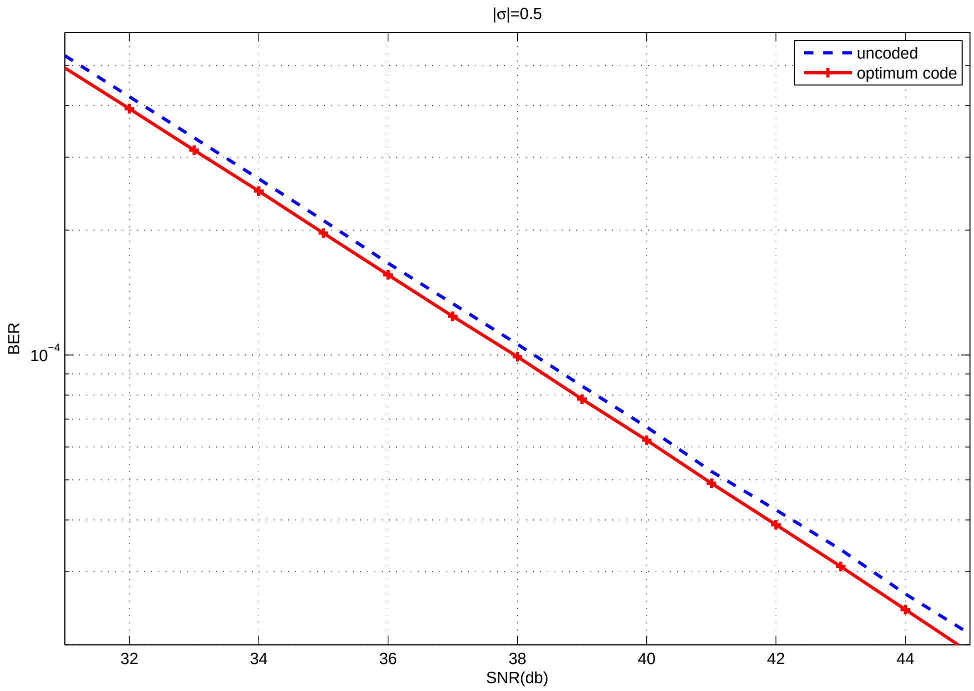


Figure 7.4: Comparison of SNR-BER when $\sigma = 0.5e^{0.5j}$

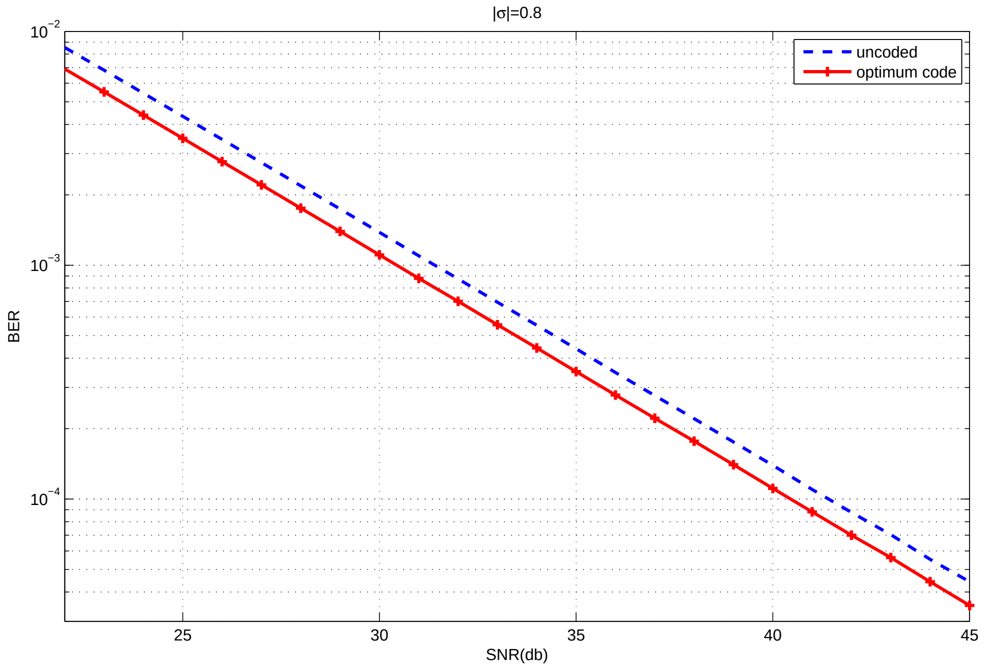


Figure 7.5: Comparison of SNR-BER when $\sigma = 0.8e^{0.5j}$

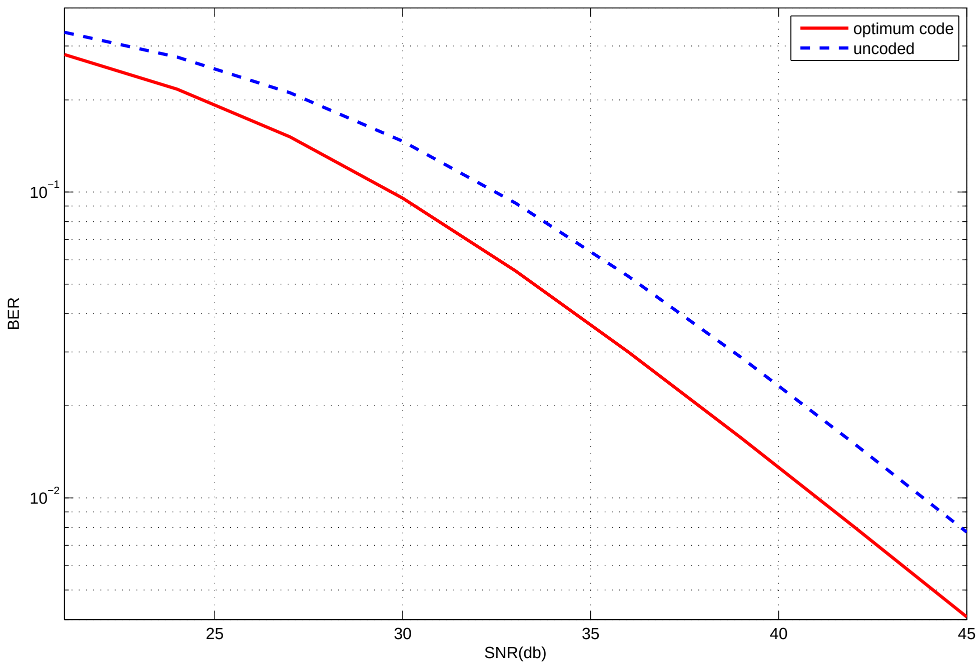


Figure 7.6: Comparison of SNR-BER when $\sigma = 0.999e^{0.5j}$

In this chapter we have applied our designed optimum code to simulations, in the last chapter we will emphasize the structure of optimum code and state the future potential work.

Chapter 8

Conclusion and Future Work

8.1 Conclusion

In this thesis we have designed optimum Linear Space Time Block Code that minimizes BER (4-QAM) when using Zero-Forcing receiver in both uncorrelated and correlated MIMO channel. The original problem is restricted to a convex optimization problem by applying the conditions of convex functions. However, the convex optimization problem which needs to find the optimum matrix contains complex structure such as Kronecker product. So we simplify it into vector form and eventually reformulate it into single variable which is much simpler than the original problem. The methodology is firstly minimize the lower bound of the objective–bit error rate and secondly achieve the minimized lower bound by satisfying all the conditions.

For a 2×2 MIMO system, by deriving the asymptotic formula of bit error rate, we find that the diversity gain is one. By theoretical analysis, numerical evaluation and simulation results, we find the optimum STBC structure Eq. (4.9) is following :

$$\mathbf{F} = \begin{pmatrix} \bar{\mathbf{F}} & 0 & \cdots & 0 \\ 0 & \bar{\mathbf{F}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \bar{\mathbf{F}} \end{pmatrix} \mathbf{V}_F^H \tag{8.1}$$

where $\mathbf{V}_F^H$ is a DFT matrix generated according to the procedures in Chapter 6.

Besides,

$$\bar{\mathbf{F}} = \mathbf{V}_\Sigma \begin{pmatrix} \sqrt{p_{\text{opt}}} & 0 \\ 0 & \sqrt{2 - p_{\text{opt}}} \end{pmatrix} \tag{8.2}$$

where $\mathbf{V}_\Sigma$ is the unitary matrix from eigendecomposition of channel covariance matrix Σ.

As for p_{opt}, we have

$$\begin{pmatrix} \sqrt{p_{\text{opt}}} & 0 \\ 0 & \sqrt{2 - p_{\text{opt}}} \end{pmatrix} = \sqrt{\frac{M}{\text{tr}\left(\mathbf{D}_{\Sigma^{-1}}^{\frac{1}{2}}\right)}} \mathbf{D}_{\Sigma^{-1}}^{\frac{1}{4}} \tag{8.3}$$

By minimizing the asymptotic formula in Chapter 5, we have found that problem of choosing p_{opt} is the same with optimum precoder design in known channels [1]. We also verify that when using the p_{opt} in Eq. (8.3), bit error rate is minimized.

Therefore, the optimum space-time block code for each symbol is trace-orthogonal to each other and power is distributed according to the The performance of optimum code is much better than the uncoded system and as channel is more correlated, the advantage is more apparent.

While our derivation has been performed by considering the 4-QAM modulation, the same technique can be extended to show that this optimum code structure is also optimum in minimizing the average asymptotic BER for a general square QAM signaling with $2b_l$ bits per symbol. Here, the BER $\tilde{P}_e(\mathbf{F})$ for the ZF detector is closely approximated by [31]

$$\tilde{P}_e(\mathbf{F}) \approx \frac{1}{2MT} \sum_{l=1}^{2MT} E_{\mathrm{H}} \left\{ \alpha_l Q \left(\sqrt{\delta_l([V_{be}]_{ll}^{-1})} \right) + \beta_l Q \left(3\sqrt{\delta_l([V_{be}]_{ll}^{-1})} \right) \right\} \tag{8.4}$$

where $\alpha_l = (2^{b_l} - 1)/b_l 2^{b_l - 1}$, $\beta_l = (2^{b_l} - 2)/b_l 2^{b_l - 1}$ and $\delta_l = (3 \cdot 2^{b_l})/(4^{b_l} - 1)$. Following a route similar to that of the optimum code of 4-QAM, we can see that the optimum code also works for square QAM.

8.2 Future Work

As for the complexity of integrals, we only solve the problem of 2×2 MIMO system. Expansion to general case $M \times N$ system can be considered in the future. Futhermore, the constellation from 4-QAM to other constellations can also be researched on, such as M-QAM.

Appendix A

Mathematical Knowledge

A.1 Kronecker Product

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is a $p \times q$ matrix, then the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is the $mp \times nq$ block matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \tag{A.1}$$

A.2 Circular Symmetric Complex Normal Distribution

If $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$ is circular complex normal, then the vector vec$[\mathbf{XY}]$ is multivariate normal with covariance structure

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \sim \mathcal{N}\Big(\begin{bmatrix} \text{Re } \mu \\ \text{Im } \mu \end{bmatrix}, \tfrac{1}{2} \begin{bmatrix} \text{Re } \mathbf{\Gamma} & -\text{Im } \mathbf{\Gamma} \\ \text{Im } \mathbf{\Gamma} & \text{Re } \mathbf{\Gamma} \end{bmatrix} \Big)$$

where $\mu = E[\mathbf{Z}]$ and $\mathbf{\Gamma} = E[\mathbf{Z}\mathbf{Z}^H]$. This is usually denoted $\mathbf{Z} \sim \mathcal{CN}(\mu, \mathbf{\Gamma})$.

A.3 Wishart Distribution

A.3.1 Definition

For complex matrix $\mathbf{H}$, where $\mathbf{H}_{N\times M}$ is $\mathcal{CN}(\mathbf{0}, \mathbf{I}_N \otimes \mathbf{\Sigma})$, then $\mathbf{H}^H\mathbf{H}$ is said to have the Wishart distribution with N degree of freedom and covariance matrix $\mathbf{\Sigma}$, which is denoted by $\mathcal{W}_M(N, \mathbf{\Sigma})$, $(N \geq M)$.

The number of degrees of freedom is the number of values in the final calculation of a statistic that are free to vary, and here degree of freedom is N.

A.3.2 Proof of $\mathbf{H}^H\mathbf{H} \sim \mathcal{W}_M(N, \mathbf{\Sigma})$

From Eq. (2.3), when $\mathbf{H}$ is correlated in receiver end, we have $\mathbf{H} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N \otimes \mathbf{R}_R)$, where $\mathbf{R}_R = \mathbf{\Sigma}$. And applying the definition of wishart distribution, $\mathbf{H}^H\mathbf{H} \sim \mathcal{W}_M(N, \mathbf{\Sigma})$ with degree of freedom N and covariance matrix $\mathbf{\Sigma}$.

A.3.3 Property

Here is a property of Wishart Distribution [23]:

For real matrix, if $\mathbf{X} \sim \mathcal{W}(n, \mathbf{Y})$, $\mathbf{Z}\mathbf{X}\mathbf{Z}^T \sim \mathcal{W}(n, \mathbf{Z}\mathbf{Y}\mathbf{Z}^T)$. It can also be generalized to complex matrix.

In our system model, we already know that $\mathbf{H}^H\mathbf{H} \sim \mathcal{W}(N, \mathbf{\Sigma})$, if we define $\tilde{\mathbf{H}} = \mathbf{H}\mathbf{\Sigma}^{-\frac{1}{2}}$, applying this property, we have

$$\begin{aligned}\tilde{\mathbf{H}}^H\tilde{\mathbf{H}} &= {\mathbf{\Sigma}^{-\frac{1}{2}}}^H\mathbf{H}^H\mathbf{H}\mathbf{\Sigma}^{-\frac{1}{2}} \\ &\sim \mathcal{W}(N, {\mathbf{\Sigma}^{-\frac{1}{2}}}^H\mathbf{\Sigma}\mathbf{\Sigma}^{-\frac{1}{2}}) \\ &\sim \mathcal{W}(N, \mathbf{I}_M)\end{aligned}$$

which means $\tilde{\mathbf{H}}$ is an i.i.d channel.

Besides, since $\hat{\mathbf{H}} = \tilde{\mathbf{H}}\mathbf{V}_{\tilde{F}}$, where $\mathbf{V}_{\tilde{F}}$ is a $M \times M$ unitary matrix. Applying the property of Wishart distribution again, we have

$$\begin{aligned}\hat{\mathbf{H}}^H\hat{\mathbf{H}} &= \mathbf{V}_{\tilde{F}}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{V}_{\tilde{F}} \\ &\sim \mathcal{W}(N, \mathbf{V}_{\tilde{F}}^H\mathbf{I}_M\mathbf{V}_{\tilde{F}}) \\ &\sim \mathcal{W}(N, \mathbf{I}_M)\end{aligned}$$

Therefore, $\hat{\mathbf{H}}$ is also an i.i.d channel now.

A.4 Pseudo-inverse

In mathematics, a pseudo-inverse of a matrix $\mathbf{A}$ is a matrix that has some properties of the inverse matrix of $\mathbf{A}$ but not necessarily all of them. The term "the pseudo-inverse" commonly means the MoorePenrose pseudo-inverse.

The purpose of constructing a pseudo-inverse is to obtain a matrix that can serve as the inverse in some sense for a wider class of matrices than invertible ones. Typically, the pseudo-inverse exists for an arbitrary matrix, and when a matrix has an inverse, then its inverse and the pseudo-inverse are the same.

Assume that $\mathbf{A}$ has full rank, then

$$\begin{array}{lllll} \mathbf{A} & N \times N & \text{Square} & \text{rank}(\mathbf{A}) = N & \Rightarrow \ \mathbf{A}^{\dagger} = \mathbf{A}^{-1} \\ \mathbf{A} & N \times M & \text{Broad} & \text{rank}(\mathbf{A}) = N & \Rightarrow \ \mathbf{A}^{\dagger} = \mathbf{A}^{H}\left(\mathbf{A}\mathbf{A}^{H}\right)^{-1} \\ \mathbf{A} & N \times M & \text{Tall} & \text{rank}(\mathbf{A}) = M & \Rightarrow \ \mathbf{A}^{\dagger} = \left(\mathbf{A}^{H}\mathbf{A}\right)^{-1}\mathbf{A}^{H} \end{array}$$

Therefore, when $\mathcal{H}$ $(NT \times MT)$ is a tall matrix, we have

$$\mathcal{H}^{\dagger} = \left(\mathcal{H}^{H}\mathcal{H}\right)^{-1}\mathcal{H}^{H}$$

A.5 Q-Function

In statistics, the Q-function, denoted by $Q(x)$, is the probability that a standard normal random variable will obtain a value larger than x.

Formally, the Q-function is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} \exp\Big(-\frac{u^2}{2}\Big)du. \tag{A.2}$$

Another form of Q-function expressed by trigonometric function is

$$Q(x) = \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}d\theta \tag{A.3}$$

Appendix B

Convex Optimization

B.1 First- and Second Order Conditions of Convex Functions

B.1.1 First Order Conditions

Suppose f is differentiable (i.e., its gradient ∇f exists at each point in $\mathbf{dom}\ f$). Then f is convex if and only if $\mathbf{dom}\ f$ is convex and

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x})$$

holds for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{dom}\ f$.

Consider the case $f : \mathbb{R} \to \mathbb{R}$: we show that a differentiable function f is convex if and only if

$$f(y) \geq f(x) + f'(x)(y - x) \tag{B.4}$$

for all x and y in **dom** f.

B.1.2 Second Order Conditions

We now assume that f is twice differentiable, that is, its Hessian or second derivative $\nabla^2 f$ exists at each point in **dom** f. Then f is convex if and only if **dom** f is convex and its Hessian is positive semidefinite: for all $\mathbf{x} \in$ **dom** f:

$$\nabla^2 f(\mathbf{x}) \geq 0 \tag{B.5}$$

For a function $f : \mathbb{R} \to \mathbb{R}$, this reduces to the simple condition $f''(x) \geq 0$ (and **dom** f convex), which means that the derivative is nondecreasing.

B.1.3 Conditions for Convex Function $f(x) = Q(\sqrt{x^{-1}})$

According to the definition of Q-function in Eq. (A.2),

$$\begin{aligned} f(x) &= Q(\sqrt{x^{-1}}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{x^{-\frac{1}{2}}}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \end{aligned}$$

therefore, the first-order and second-order derivatives are respectively

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^{-1}\right) x^{-\frac{3}{2}} \\ f''(x) &= \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^{-1}\right) x^{-\frac{5}{2}} \left(\frac{1}{2}x^{-1} - \frac{3}{2}\right) \end{aligned}$$

Here it is easy to find that $f'(x) > 0$ all the time, so that $f(x)$ is monotonically increasing with x.

If we want $f(x)$ to be a convex function, apply the second condition in Eq. (B.5):

- **dom** f is set $\mathbb{R}_+$, which means positive real number set and is a convex set.

- Hessian of f is positive semidefinite:

$$\begin{aligned} f''(x) \geq 0 &\Longrightarrow \frac{1}{2}x^{-1} - \frac{3}{2} \\ &\Longrightarrow 0 < x \leq \frac{1}{3} \end{aligned}$$

Therefore, when $0 < x \leq \frac{1}{3}$, $f(x) = Q(\sqrt{x^{-1}})$ is convex.

B.2 Jensen's Inequality

Jensen's inequality [25] is expressed as

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

It is easily extended to convex combinations of more than two points: If f is convex, $x_1, \cdots, x_k \in \textbf{dom}\ f$, and $\theta_1, \cdots, \theta_k \geq 0$ with $\theta_1 + \cdots + \theta_k = 1$, then

$$f(\theta_1 x_1 + \cdots + \theta_k x_k) \leq \theta_1 f(x_1) + \cdots + \theta_k f(x_k)$$

In the most general case we can take any probability measure with support in **dom** f. If $\mathbf{x}$ is a random variable such that $\mathbf{x} \in \textbf{dom} f$ with probability one, and f is convex, then we have

$$f(\mathrm{E}\mathbf{x}) \leq \mathrm{E} f(\mathbf{x})$$

provided the expectations exist. Equality holds if and only if all x selected from $\mathbf{x}$ are equal.

Appendix C

Matrix Computations

C.1 QR Decomposition

Any real square matrix $\mathbf{X}$ may be decomposed as

$$\mathbf{X} = \mathbf{Q}_X \mathbf{R}_X,$$

where $\mathbf{Q}_X$ is an orthogonal matrix (its columns are orthogonal unit vectors meaning $\mathbf{Q}_X^T \mathbf{Q}_X = \mathbf{I}$) and $\mathbf{R}_X$ is an upper triangular matrix (also called right triangular matrix). If $\mathbf{X}$ is nonsingular, then the factorization is unique if we require that the diagonal elements of $\mathbf{Q}_X$ are positive. This generalizes to a complex square matrix $\mathbf{X}$ and a unitary matrix $\mathbf{Q}_X$.

C.2 Chi-Square Distribution

If $Z_1, ..., Z_k$ are independent, standard normal random variables, then the sum of their squares $Q = \sum_{i=1}^{k} Z_i^2$ is distributed according to the chi-square distribution with k degrees of freedom. This is usually denoted as $Q \sim \chi_k^2$.

The chi-square distribution has one parameter k: a positive integer that specifies the number of degrees of freedom (i.e. the number of Z_is).

The probability density function (pdf) of the chi-square distribution is

$$f(x;\, k) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$$

where $\Gamma(k/2)$ denotes the Gamma function. If n is a positive integer, we have

$$\Gamma(n) = (n-1)!$$

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}$$

C.3 Bartlett Decomposition

For real matrix, let $\mathbf{X}$ be $\mathcal{W}(n, \mathbf{I}_m)$, where $n \geq m$, and put $\mathbf{X} = \mathbf{R}^T\mathbf{R}$, where $\mathbf{R}$ is an upper triangular $m \times m$ matrix with positive diagonal elements, then the elements $r_{ij}(1 \leq i \leq j \leq m)$ of $\mathbf{R}$ are all independent, r_{ii}^2 is $\mathcal{X}_{n-i+1}^2$ and r_{ij} is $\mathcal{N}(0,1)$ $(1 \leq i < j \leq m)$. It can also be generalized to complex matrix.

C.4 Probability Density Function of x_1, x_2, x_3

Because we have $\hat{\mathbf{H}}^H\hat{\mathbf{H}} = \mathbf{R}_{\hat{H}}^H\mathbf{R}_{\hat{H}}$, where $\mathbf{R}_{\hat{H}}^H$ is an upper triangular 2×2 matrix with positive diagonal elements, which satisfy the conditions of Bartlett decomposition. Therefore, we have following conclusion:

a) the elements $r_{ij}(1 \le i \le j \le 2)$ of $\mathbf{R}_{\hat{H}}$ are all independent and complex numbers.

b) $[\mathrm{Re}(r_{11})]^2$ and $[\mathrm{Im}(r_{11})]^2$ are both $\mathcal{X}^2_{2-1+1} = \mathcal{X}^2_2$, so that applying definition of chi-square distribution,

$$x_1 = |r_{11}|^2 = [\mathrm{Re}(r_{11})]^2 + [\mathrm{Im}(r_{11})]^2 \sim \chi^2_4$$

The probability density function of x_1 is

$$\begin{aligned} p(x_1) &\sim \chi^2_4 \\ &= c_1 x_1^{4/2-1} e^{-x_1} \\ &= c_1 x_1 e^{-x_1} \end{aligned}$$

where c_1 is a constant, to calculate c_1, we use integral that

$$
\begin{aligned}
&\int_0^\infty p(x_1)dx_1 = 1 \\
\Rightarrow &\int_0^\infty c_1 x_1 e^{-x_1} dx_1 = 1 \\
\Rightarrow &c_1 \int_0^\infty -x_1 de^{-x_1} = 1 \\
\Rightarrow &- c_1[x_1 e^{-x_1} |_0^\infty - \int_0^\infty e^{-x_1} dx_1] = 1 \\
\Rightarrow &- c_1[0 + e^{-x_1} |_0^\infty] = 1 \\
\Rightarrow &c_1 = 1
\end{aligned}
$$

Therefore, the probability density function of x_1 is

$$p(x_1) = x_1 e^{-x_1}$$

c) $[\text{Re}(r_{22})]^2$ and $\text{Im}(r_{22})^2$ are both $\mathcal{X}^2_{2-2+1} = \mathcal{X}^1_2$, so that applying definition of chi-square distribution,

$$x_2 = |r_{22}|^2 = [\text{Re}(r_{22})]^2 + [\text{Im}(r_{22})]^2 \sim \chi^2_2$$

The probability density function of x_2 is

$$
\begin{aligned}
p(x_2) &\sim \chi^2_2 \\
&= c_2 x_2^{2/2-1} e^{-x_2} \\
&= c_2 e^{-x_2}
\end{aligned}
$$

where c_1 is a constant, to calculate c_2, we use integral that

$$\begin{aligned}
&\int_0^\infty p(x_2)dx_2 = 1 \\
\Rightarrow &\int_0^\infty c_2 e^{-x_2} dx_2 = 1 \\
\Rightarrow &- c_2[e^{-x_2} |_0^\infty] = 1 \\
\Rightarrow &c_2 = 1
\end{aligned}$$

Therefore, the probability density function of x_2 is

$$p(x_2) = e^{-x_2}$$

d) $\mathrm{Re}(r_{12})$ and $\mathrm{Im}(r_{12})$ are both $\mathcal{N}(0,1)$ $(1 \leq i < j \leq m)$, so that applying definition of chi-square distribution,

$$x_3 = |r_{12}|^2 = \mathrm{Re}(r_{12})^2 + \mathrm{Im}(r_{12})^2 \sim \chi_2^2$$

Similar to x_2, the probability density function of x_3 is

$$p(x_3) = e^{-x_3}$$

C.5 Eigenvalues and Eigenvectors

The eigenvectors $\mathbf{v}$ and eigenvalues λ are the ones satisfying

$$\mathbf{A}\mathbf{v}_{Ai} = \lambda_{Ai}\mathbf{v}_{Ai}$$

$$\mathbf{A}\mathbf{V}_A = \mathbf{V}_A\mathbf{D}_A$$

where $(\mathbf{D})_{Aij} = \delta_{ij}\lambda_{Ai}$ and the columns of $\mathbf{V}_A$ are the vectors $\mathbf{v}_{Ai}$.

Assume A is symmetric, then

$$\mathbf{V}_A\mathbf{V}_A^T = I$$

i.e. $\mathbf{V}_A$ is orthogonal and $\lambda_{Ai} \in \mathbb{R}$, i.e. λ_{Ai} is real.

C.6 Singular Value Decomposition

Any $n \times m$ matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{U}_A\mathbf{D}_A\mathbf{V}_A^T$$

where

$$\begin{aligned} \mathbf{U}_A &= \text{eigenvectors of } \mathbf{A}\mathbf{A}^T & n \times n \\ \mathbf{D}_A &= \sqrt{(\text{diag}(\text{eig}(\mathbf{A}\mathbf{A}^T)))} & n \times m \\ \mathbf{V}_A &= \text{eigenvectors of } \mathbf{A}^T\mathbf{A} & m \times m \end{aligned}$$

Assume $\mathbf{A}$ to be $n \times n$ and symmetric, then

$$\mathbf{A} = \mathbf{V}_A\mathbf{D}_A\mathbf{V}_A^T$$

where $\mathbf{D}_A$ is diagonal with the eigenvalues of $\mathbf{A}$, and $\mathbf{V}_A$ is orthogonal and the eigenvectors of $\mathbf{A}$. This is also called eigendecomposition.

C.7 Eigenvalues of Σ

Σ is a 2×2 matrix:

$$\Sigma = \begin{pmatrix} 1 & \sigma \\ \sigma^* & 1 \end{pmatrix} \tag{C.6}$$

The eigenvectors $\mathbf{v}_i$ and eigenvalues λ_i of Σ are the ones satisfying

$$\boldsymbol{\Sigma}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

where $i = 1, 2$ in such case.

Therefore,

$$\begin{aligned}
&(\boldsymbol{\Sigma} - \lambda_i \mathbf{I})\mathbf{v}_i = \mathbf{0} \\
\Rightarrow &\boldsymbol{\Sigma} - \lambda_i \mathbf{I} = \mathbf{0} \\
\Rightarrow &\begin{pmatrix} 1-\lambda_i & \sigma \\ \sigma^* & 1-\lambda_i \end{pmatrix} = \mathbf{0} \\
\Rightarrow &\det \begin{pmatrix} 1-\lambda_i & \sigma \\ \sigma^* & 1-\lambda_i \end{pmatrix} = 0 \\
\Rightarrow &(1-\lambda_i)(1-\lambda_i) - \sigma\sigma^* = 0 \\
\Rightarrow &(1-\lambda_i)^2 - |\sigma|^2 = 0 \\
\Rightarrow &1 - \lambda_i = \pm|\sigma| \\
\Rightarrow &\lambda_i = 1 \pm |\sigma|
\end{aligned}$$

So the two eigenvalues of Σ are

$$\begin{cases} \lambda_1 = 1 + |\sigma| \\ \lambda_2 = 1 - |\sigma| \end{cases}$$

C.8 Structure of Optimum Code F

We already know that $\mathbf{A} = \mathbf{F}\mathbf{F}^H$ is block diagonal and with each matrix on the diagonal is $\bar{\mathbf{F}}\bar{\mathbf{F}}^H$.

Because $\mathbf{F}\mathbf{F}^H$ is symmetric, applying eigendecomposition, we define

$$\mathbf{F}\mathbf{F}^H = \mathbf{V}_A \mathbf{D}_A \mathbf{V}_A^H \tag{C.7}$$

where $\mathbf{V}_A = \mathbf{I}$ and $\mathbf{D}_A$ is block diagonal and with each matrix on the diagonal is $\bar{\mathbf{F}}\bar{\mathbf{F}}^H$.

Applying singular value decomposition on $\mathbf{F}$, we define

$$\mathbf{F} = \mathbf{U}_F \mathbf{D}_F \mathbf{V}_F^H$$

Therefore,

$$\begin{aligned} \mathbf{F}\mathbf{F}^H &= \mathbf{U}_F \mathbf{D}_F \mathbf{V}_F^H \mathbf{V}_F \mathbf{D}_F^H \mathbf{U}_F^H \\ &= \mathbf{U}_F \mathbf{D}_F \mathbf{D}_F^H \mathbf{U}_F^H \end{aligned}$$

Compared with Eq. (C.7), $\mathbf{U}_F = \mathbf{I}$ and $\mathbf{D}_F \mathbf{D}_F^H = \mathbf{D}_A$. Therefore, $\mathbf{D}_F$ is block diagonal with each matrix on the diagonal is $\bar{\mathbf{F}}$.

Above all,

$$\mathbf{F} = \mathbf{D}_F \mathbf{V}_F^H$$

Expressing the equation more vividly, we have

$$\mathbf{F} = \begin{pmatrix} \bar{\mathbf{F}} & 0 & \cdots & 0 \\ 0 & \bar{\mathbf{F}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \bar{\mathbf{F}} \end{pmatrix} \mathbf{V}_F^H \tag{C.8}$$

where $\mathbf{V}_F^H$ is a unitary matrix which can be generated later.

Appendix D

Calculation of Distribution Function $F_x(t)$

$$
\begin{aligned}
F_x(t) &= P\{x \le t\} \\
&= P\left(\left(\frac{x_2 + x_3}{x_1 x_2}\right) d_1^{-1} + \frac{d_2^{-1}}{x_2} \le t\right) \\
&= P\left(x_3 \le \frac{x_1 x_2 t - x_2 d_1^{-1} - x_1 d_2^{-1}}{d_1^{-1}}\right) \\
&= \int_{x_1} \int_{x_2} \int_{x_3} p(x_1) p(x_2) p(x_3) dx_3 dx_2 dx_1
\end{aligned} \tag{D.9}
$$

First we find out the boundaries of variables x_1, x_2 and x_3 respectively.

Because we know that

$$
\begin{aligned}
&x_1 \geq 0 \\
&x_2 \geq 0 \\
&0 \leq x_3 \leq \frac{x_1 x_2 t - x_2 d_1^{-1} - x_1 d_2^{-1}}{d_1^{-1}} t > 0
\end{aligned}
$$

We also know that eigenvalues of $\tilde{F}\tilde{F}^H$ have the range that

$$
\begin{aligned}
d_1 \geq 0 \\
d_2 \geq 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{x_1 x_2 t - x_2 d_1^{-1} - x_1 d_2^{-1}}{d_1^{-1}} \geq 0 \\
\Rightarrow & x_1 x_2 t - x_2 d_1^{-1} - x_1 d_2^{-1} \geq 0 \\
\Rightarrow & (x_1 t - d_1^{-1}) x_2 \geq x_1 d_2^{-1} \\
\Rightarrow & x_1 \geq \frac{d_1^{-1}}{t} \\
& x_2 \geq \frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}} \qquad \text{(D.10)}
\end{aligned}
$$

$$\text{(D.11)}$$

Substituting the boundaries of x_1, x_2 and x_3 into Eq. (D.9), we have

$$F_x(t) = \int_{\frac{d_1^{-1}}{t}}^{\infty} \int_{\frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}}}^{\infty} \int_0^{\frac{x_1 x_2 t - x_2 d_1^{-1} - x_1 d_2^{-1}}{d_1^{-1}}} p(x_1)p(x_2)p(x_3)dx_3dx_2dx_1 \tag{D.12}$$

We already expressed the probability density functions of x_1, x_2 and x_3 as follows:

$$p(x_1) = x_1 e^{-x_1}$$

$$p(x_2) = e^{-x_2}$$

$$p(x_3) = e^{-x_3}$$

Therefore,

$$
\begin{aligned}
F_x(t) &= \int_{\frac{d_1^{-1}}{t}}^{\infty} \int_{\frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}}}^{\infty} \left(\int_0^{\frac{x_1 x_2 t - x_2 d_1^{-1} - x_1 d_2^{-1}}{d_1^{-1}}} e^{-x_3} dx_3 \right) e^{-x_2} dx_2 \left(x_1 e^{-x_1} \right) dx_1 \\
&= \int_{\frac{d_1^{-1}}{t}}^{\infty} \int_{\frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}}}^{\infty} \left(-e^{-x_3} \Big|_0^{\frac{x_1 x_2 t - x_2 d_1^{-1} - x_1 d_2^{-1}}{d_1^{-1}}} \right) e^{-x_2} dx_2 \left(x_1 e^{-x_1} \right) dx_1 \\
&= \int_{\frac{d_1^{-1}}{t}}^{\infty} \int_{\frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}}}^{\infty} \left(1 - e^{-\frac{x_1 x_2 t - x_2 d_1^{-1} - x_1 d_2^{-1}}{d_1^{-1}}} \right) e^{-x_2} dx_2 \left(x_1 e^{-x_1} \right) dx_1 \\
&= \int_{\frac{d_1^{-1}}{t}}^{\infty} \int_{\frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}}}^{\infty} e^{-x_2} dx_2 \left(x_1 e^{-x_1} \right) dx_1 \\
&+ \int_{\frac{d_1^{-1}}{t}}^{\infty} \int_{\frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}}}^{\infty} \left(-e^{-\frac{x_1 x_2 t - x_1 d_2^{-1}}{d_1^{-1}}} \right) dx_2 \left(x_1 e^{-x_1} \right) dx_1 \\
&= \int_{\frac{d_1^{-1}}{t}}^{\infty} \left[-e^{-x_2} \Big|_{\frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}}}^{\infty} \right] \left(x_1 e^{-x_1} \right) dx_1 \\
&+ \int_{\frac{d_1^{-1}}{t}}^{\infty} \frac{d_1^{-1}}{x_1 t} \left[e^{-\frac{x_1 x_2 t}{d_1^{-1}}} \Big|_{\frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}}}^{\infty} \right] e^{\frac{x_1 d_2^{-1}}{d_1^{-1}}} \left(x_1 e^{-x_1} \right) dx_1 \\
&= \int_{\frac{d_1^{-1}}{t}}^{\infty} e^{-\frac{x_1 d_2^{-1}}{x_1 t - d_1^{-1}}} x_1 e^{-x_1} dx_1 \\
&- \frac{d_1^{-1}}{t} \int_{\frac{d_1^{-1}}{t}}^{\infty} e^{-x_1 \left(-\frac{d_2^{-1}}{d_1^{-1}} + 1 + \frac{t x_1 d_2^{-1}}{d_1^{-1}(x_1 t - d_1^{-1})} \right)} dx_1
\end{aligned} \tag{D.13}
$$

Because this integral is hard to solve, we will keep this form and simplify it by defining $u = x_1 t - d_1^{-1}$, and then we have

$$\begin{aligned} u &= x_1 t - d_1^{-1} \\ &\geq \frac{d_1^{-1}}{t} t - d_1^{-1} \\ &\geq 0 \end{aligned}$$

We also have

$$\begin{aligned} x_1 &= \frac{u + d_1^{-1}}{t} \\ dx_1 &= \frac{1}{t} du \end{aligned}$$

Therefore, the distribution function $F_x(t)$ becomes

$$\begin{aligned} F_x(t) &= \int_0^\infty \frac{u + d_1^{-1}}{t^2} e^{-\frac{(u+d_1^{-1})(u+d_2^{-1})}{tu}} du - \int_0^\infty \frac{d_1^{-1}}{t^2} e^{-\frac{(u+d_1^{-1})(u+d_2^{-1})}{tu}} du \\ &= \int_0^\infty \frac{u}{t^2} e^{-\frac{(u+d_1^{-1})(u+d_2^{-1})}{tu}} du \end{aligned} \tag{D.14}$$

Appendix E

Integral Calculation of $P_e(\mathbf{F})$

From Eq. (4.24), the averaged bit error rate is expressed as

$$\begin{aligned} P_e(\mathbf{F}) &\geq \mathrm{E}_x \left\{ Q\left(\sqrt{\frac{\rho}{x}}\right) \right\} \\ &= \int_0^{\frac{\rho}{3}} p(x) Q\left(\sqrt{\frac{\rho}{x}}\right) dx \end{aligned} \tag{E.15}$$

If we let $w = \frac{1}{x}$, the boundary of w will be $w \geq \frac{3}{\rho}$ and

$$dx = \frac{-1}{w^2} dw$$

We know that there is an integral of u inside $p(x)$, therefore, after replacing the

x with w and substituting trigonometric form of Q-function, we have

$$P_e(\mathbf{F}) \geq \int_{\infty}^{\frac{3}{\rho}} p(\frac{1}{w})Q\left(\sqrt{\rho w}\right)\left(-\frac{1}{w^2}\right)dw \tag{E.16}$$

$$= \int_{\frac{3}{\rho}}^{\infty} \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} e^{\left(-\frac{\rho w}{2sin^2\theta}\right)} p(\frac{1}{w})\left(\frac{1}{w^2}\right)d\theta dw$$

$$= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}}\int_{\frac{3}{\rho}}^{\infty}\int_{0}^{\infty} f(u,w,\theta)dudwd\theta \tag{E.17}$$

Now we can see in this formula there are three layers of integral altogether, so first we must consider the order of integration.

According to Fubini's theorem, if we have

$$\int_a^b\int_c^d |f(x,y)|dydx < \infty$$

Then the two integrals have same finite values, i.e.

$$\int_a^b\int_c^d f(x,y)dxdy = \int_a^b\int_c^d f(x,y)dydx$$

As a consequence it allows the order of integration to be changed.

To prove that we have finite integral, we have to use the upper bound of Q-function:

$$Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}$$

From Eq. (E.16), we have

$$P_e(\mathbf{F}) = \int_{\frac{3}{\rho}}^{\infty} p(\frac{1}{w}) Q\left(\sqrt{\rho w}\right)\left(\frac{1}{w^2}\right) dw$$
$$\leq \int_{\frac{3}{\rho}}^{\infty} p(\frac{1}{w}) \frac{1}{2} e^{-\frac{\rho w}{2}} \left(\frac{1}{w^2}\right) dw$$

Because $p(\frac{1}{w})$ is probability density function with value between 0 and 1. And e^{-x} also has values between 0 and 1 when $x > 0$. Therefore,

$$P_e(\mathbf{F}) \leq \int_{\frac{3}{\rho}}^{\infty} \frac{1}{2}\left(\frac{1}{w^2}\right) dw \tag{E.18}$$
$$= -\frac{1}{2w} \mid_{\frac{3}{\rho}}^{\infty} = \frac{\rho}{6} < \infty \tag{E.19}$$

which means Fubini's theorem has been satisfied, and the order of integration can be changed freely.

So (4.25) can be written as

$$\begin{aligned} P_e(\mathbf{F}) &= \int_0^{\frac{\pi}{2}} \int_0^{\infty} \int_{\frac{3}{\rho}}^{\infty} f(u, w, \theta) dw du d\theta \\ &= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \int_{\frac{3}{\rho}}^{\infty} e^{-\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right] w} \left[-2wu + (u+d_1^{-1})(u+d_2^{-1})w^2\right] dw du d\theta \\ &= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \left\{ \int_{\frac{3}{\rho}}^{\infty} -2uwe^{-\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right] w} dw \right. \\ &\left. + \int_{\frac{3}{\rho}}^{\infty} (u+d_1^{-1})(u+d_2^{-1}) w^2 e^{-\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right] w} dw \right\} du d\theta \end{aligned} \tag{E.20}$$

Because we have

$$\int_{\frac{3}{\rho}}^{\infty} bwe^{-aw} dw = e^{-\frac{3a}{\rho}} \left[\frac{3b}{\rho a} + \frac{b}{a^2}\right] \tag{E.21}$$

and

$$\int_{\frac{3}{\rho}}^{\infty} bw^2 e^{-aw} dw = e^{-\frac{3a}{\rho}} \left[\frac{9b}{\rho^2 a} + \frac{6b}{\rho a^2} + \frac{2b}{a^3} \right] \tag{E.22}$$

So (E.20) becomes

$$\begin{aligned}
P_e(\mathbf{F}) = & \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-\frac{3}{\rho}\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]} \left\{ \frac{-6u}{\rho\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]} \right. \\
& + \frac{-2u}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]^2} + \frac{9(u+d_1^{-1})(u+d_2^{-1})}{\rho^2\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]} \\
& \left. + \frac{6(u+d_1^{-1})(u+d_2^{-1})}{\rho\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]^2} + \frac{2(u+d_1^{-1})(u+d_2^{-1})}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]^3} \right\} du d\theta \\
= & \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-\frac{3}{\rho}\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]} \left\{ \frac{9(u+d_1^{-1})(u+d_2^{-1})}{\rho^2\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]} \right. \\
& + \left[\frac{-6u}{\rho\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]} + \frac{6(u+d_1^{-1})(u+d_2^{-1})}{\rho\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]^2} \right] \\
& \left. + \left[\frac{-2u}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]^2} + \frac{2(u+d_1^{-1})(u+d_2^{-1})}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]^3} \right] \right\} du d\theta \\
= & \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-\frac{3}{\rho}\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]} \left\{ \frac{9(u+d_1^{-1})(u+d_2^{-1})}{\rho^2\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]} \right. \\
& \left. + \frac{\frac{-3u}{\sin^2\theta}}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]^2} + \frac{\frac{-\rho u}{\sin^2\theta}}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u} + \frac{\rho}{2\sin^2\theta}\right]^3} \right\} du d\theta
\end{aligned} \tag{E.23}$$

Now integration with variable w has been finished, and nextly we will integrate θ.

If we define $\frac{1}{\sin^2\theta} = 1 + \cot^2\theta$, Eq. (E.23) becomes

$$P_e(\mathbf{F}) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-\frac{3}{\rho}\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u}+\frac{\rho}{2}(1+\cot^2\theta)\right]}\left\{\frac{9(u+d_1^{-1})(u+d_2^{-1})}{\rho^2\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u}+\frac{\rho}{2}(1+\cot^2\theta)\right]}\right.$$
$$\left.+\frac{-3u(1+\cot^2\theta)}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u}+\frac{\rho}{2}(1+\cot^2\theta)\right]^2}+\frac{-\rho u(1+\cot^2\theta)}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u}+\frac{\rho}{2}(1+\cot^2\theta)\right]^3}\right\}dud\theta \tag{E.24}$$

Let $\cot\theta = t$, now t is from ∞ to 0 when θ is from 0 to $\frac{\pi}{2}$ and we also have

$$\begin{aligned}
dt &= d\cot\theta \\
&= d\frac{\cos\theta}{\sin\theta} \\
&= \frac{-\sin\theta\sin\theta - \cos\theta\cos\theta}{\sin\theta^2}d\theta \\
&= -\frac{1}{\sin\theta^2}d\theta \\
&= -(1+\cot^2\theta)d\theta \\
&= -(1+t^2)d\theta \\
\Rightarrow d\theta &= -\frac{1}{1+t^2}dt
\end{aligned}$$

Therefore, substitute this into Eq. (E.24), we have

$$
\begin{aligned}
P_e(\mathbf{F}) = \frac{1}{\pi}\int_0^\infty \int_0^\infty e^{-\frac{3}{\rho}\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u}+\frac{\rho}{2}(1+t^2)\right]} \Bigg\{ & \frac{9(u+d_1^{-1})(u+d_2^{-1})}{(1+t^2)\rho^2\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u}+\frac{\rho}{2}(1+t^2)\right]} \\
&+ \frac{-3u}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u}+\frac{\rho}{2}(1+t^2)\right]^2} + \frac{-\rho u}{\left[\frac{(u+d_1^{-1})(u+d_2^{-1})}{u}+\frac{\rho}{2}(1+t^2)\right]^3}\Bigg\} dtdu \\
&= \frac{1}{\pi}\int_0^\infty e^{-\frac{3}{2}a}\int_0^\infty e^{-\frac{3}{2}t^2}\left[\frac{18(u+d_1^{-1})(u+d_2^{-1})}{\rho^3}\frac{1}{(1+t^2)(a+t^2)}\right. \\
&\left. - \frac{12u}{\rho^2}\frac{1}{(a+t^2)^2} - \frac{8u}{\rho^2}\frac{1}{(a+t^2)^3}\right] dtdu \\
&= \frac{1}{\pi}\int_0^\infty e^{-\frac{3}{2}a}\Bigg\{\frac{18(u+d_1^{-1})(u+d_2^{-1})}{(a-1)\rho^3}\Bigg[\Bigg[\underbrace{\int_0^\infty \frac{e^{-\frac{3}{2}t^2}}{1+t^2}dt}_{A}\Bigg] \\
&- \Bigg[\underbrace{\int_0^\infty \frac{e^{-\frac{3}{2}t^2}}{a+t^2}dt}_{B}\Bigg]\Bigg] - \frac{12u}{\rho^2}\Bigg[\underbrace{\int_0^\infty \frac{e^{-\frac{3}{2}t^2}}{(a+t^2)^2}dt}_{C}\Bigg] - \frac{8u}{\rho^2}\Bigg[\underbrace{\int_0^\infty \frac{e^{-\frac{3}{2}t^2}}{(a+t^2)^3}dt}_{D}\Bigg]\Bigg\} du
\end{aligned}
\tag{E.25}
$$

where $a = \frac{2(u+d_1^{-1})(u+d_2^{-1})}{\rho u} + 1$.

We have

$$
\begin{aligned}
A &= \int_0^\infty \frac{e^{-\frac{3}{2}t^2}}{a+t^2}dt \\
&= \frac{\pi}{2}\mathrm{erfc}(\frac{\sqrt{6}}{2})e^{\frac{3}{2}}
\end{aligned}
$$

$$
\begin{aligned}
B &= \int_0^\infty \frac{e^{-\frac{3}{2}t^2}}{a+t^2}dt \\
&= \frac{\pi}{2\sqrt{a}}\mathrm{erfc}(\frac{\sqrt{6a}}{2})e^{\frac{3a}{2}}
\end{aligned}
$$

$$
\begin{aligned}
C &= \int_0^\infty \frac{e^{-\frac{3}{2}t^2}}{(a+t^2)^2}dt \\
&= \frac{1}{2a^{\frac{3}{2}}}\left[\frac{1}{2}\pi(1-3a)e^{\frac{3a}{2}}\mathrm{erfc}(\frac{1}{2}\sqrt{6a}) + \frac{1}{2}\sqrt{6a\pi}\right]
\end{aligned}
$$

$$
\begin{aligned}
D &= \int_0^\infty \frac{e^{-\frac{3}{2}t^2}}{(a+t^2)^3}dt \\
&= \frac{\sqrt{6}}{12a^3}\left[-\frac{1}{4}\sqrt{\pi}a(-9+9a) + \frac{3}{8}\sqrt{6a}\pi(-2a+1+3a^2)e^{\frac{3a}{2}}\mathrm{erfc}(\frac{\sqrt{6a}}{2})\right]
\end{aligned}
$$

After four separate integration and simplification, the complicated integral becomes:

$$
P_e(\mathbf{F}) = \int_0^\infty \frac{u}{\rho^2}\left[\frac{9}{2}e^{-\frac{3}{2}(a-1)}\mathrm{erfc}(\frac{\sqrt{6}}{2}) - \sqrt{\frac{27}{2\pi}}(a^{-2}+a^{-1})e^{-\frac{3}{2}a} - \frac{3}{2}a^{-\frac{5}{2}}\mathrm{erfc}(\frac{\sqrt{6a}}{2})\right] du \tag{E.26}
$$

where $a = \frac{2(u+d_1^{-1})(u+d_2^{-1})}{\rho u} + 1$ and error function $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-m^2}dm$.

Appendix F

Asymptotic Formula

From Eq. (4.25), the averaged bit error rate is expressed as

$$
\begin{aligned}
P_e(\mathbf{F}) &= \int_0^\infty \frac{u}{\rho^2}\left[\frac{9}{2}e^{-\frac{3}{2}(a-1)}\mathrm{erfc}(\frac{\sqrt{6}}{2}) - \sqrt{\frac{27}{2\pi}}(a^{-2}+a^{-1})e^{-\frac{3}{2}a} - \frac{3}{2}a^{-\frac{5}{2}}\mathrm{erfc}(\frac{\sqrt{6a}}{2})\right]du \\
&= \underbrace{\int_0^\infty \frac{u}{\rho^2}\frac{9}{2}e^{-\frac{3}{2}(a-1)}\mathrm{erfc}(\frac{\sqrt{6}}{2})du}_{A} \\
&- \underbrace{\int_0^\infty \frac{u}{\rho^2}\sqrt{\frac{27}{2\pi}}a^{-2}e^{-\frac{3}{2}a}du}_{B} \\
&- \underbrace{\int_0^\infty \frac{u}{\rho^2}\sqrt{\frac{27}{2\pi}}a^{-1}e^{-\frac{3}{2}a}du}_{C} \\
&- \underbrace{\int_0^\infty \frac{u}{\rho^2}\frac{3}{2}a^{-\frac{5}{2}}\mathrm{erfc}(\frac{\sqrt{6a}}{2})du}_{D} \\
&= A - B - C - D \qquad \text{(F.27)}
\end{aligned}
$$

where A, B, C and D are four separate integrals which will be calculated later on.

$$\begin{aligned} a &= \frac{2(u+d_1^{-1})(u+d_2^{-1})}{\rho u} + 1 \\ &= \frac{2u^2 + (2d_1^{-1} + 2d_2^{-1} + \rho)u + 2d_1^{-1}d_2^{-1}}{\rho u} \end{aligned} \tag{F.28}$$

For simplification, we define

$$\begin{cases} M = d_1^{-1} + d_2^{-1} \\ N = d_1^{-1}d_2^{-1} \end{cases} \tag{F.29}$$

Therefore, Eq. (F.28) becomes

$$a = \frac{2u^2 + (2M + \rho)u + 2N}{\rho u} \tag{F.30}$$

and

$$a - 1 = \frac{2u^2 + 2Mu + 2N}{\rho u} \tag{F.31}$$

Now we try to calculate the four integrals A, B, C and D in Eq. (F.27) to find the coding gain and diversity gain respectively.

a) **Integral** A:

$$
\begin{aligned}
A &= \int_0^\infty \frac{u}{\rho^2}\frac{9}{2}e^{-\frac{3}{2}(a-1)}\mathrm{erfc}(\frac{\sqrt{6}}{2})du \\
&= \int_0^\infty \frac{u}{\rho^2}\frac{9}{2}e^{-\frac{3}{2}(\frac{2u^2+2Mu+2N}{\rho u})}\mathrm{erfc}(\frac{\sqrt{6}}{2})du \\
&= \frac{9}{2}\mathrm{erfc}(\frac{\sqrt{6}}{2})\int_0^\infty \frac{u}{\rho^2}e^{-\frac{3u}{\rho}}e^{-\frac{3N}{\rho u}}e^{-\frac{3M}{\rho}}du
\end{aligned}
$$

Lemma 1: Exponential Series

$$
e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}
$$

therefore

$$
\begin{cases}
e^{-\frac{3M}{\rho}} = 1 - \frac{3M}{\rho} + O(\rho^{-2}) \\
e^{-\frac{3u}{\rho}} = 1 - \frac{3u}{\rho} + O(\rho^{-2}) \quad 0 < u < \delta \\
e^{-\frac{3N}{\rho u}} = 1 - \frac{3N}{\rho u} + O(\rho^{-2}) \quad u > \delta
\end{cases}
$$

when $\rho \to \infty$.

Applying Lemma 1 on A, we have

$$
A = \underbrace{[1 - \frac{3M}{\rho} + O(\rho^{-2})]\frac{9}{2}\mathrm{erfc}(\frac{\sqrt{6}}{2})}_{CONST}\int_0^\infty \frac{u}{\rho^2}e^{-\frac{3u}{\rho}}e^{-\frac{3N}{\rho u}}du
$$

Furthermore the integral part is divided into two parts A_1 and A_2:

$$\int_0^\infty \frac{u}{\rho^2} e^{-\frac{3u}{\rho}} e^{-\frac{3N}{\rho u}} du = \underbrace{\int_0^\delta \frac{u}{\rho^2} e^{-\frac{3u}{\rho}} e^{-\frac{3N}{\rho u}} du}_{A_1} + \underbrace{\int_\delta^\infty \frac{u}{\rho^2} e^{-\frac{3u}{\rho}} e^{-\frac{3N}{\rho u}} du}_{A_2} \tag{F.32}$$

where δ is a trivial number $0 < \delta \leq 1$.

Applying Lemma 1 on A_1, we obtain

$$\begin{aligned} A_1 &= \int_0^\delta \left(1 - \frac{3u}{\rho} + O(\rho^{-2})\right) \frac{u}{\rho^2} e^{-\frac{3N}{\rho u}} du \\ &= \int_0^\delta \left(1 - \frac{3u}{\rho}\right) \frac{u}{\rho^2} e^{-\frac{3N}{\rho u}} du + O(\rho^{-2}) \end{aligned}$$

Replace u with $t = \frac{3N}{\rho u}$, so that $u = \frac{3N}{\rho t}$ and $du = \frac{3N}{\rho}(-1/t^2)dt$, A_1 changes into

$$\begin{aligned} A_1 &= \int_0^\delta \left(1 - \frac{3u}{\rho}\right) \frac{u}{\rho^2} e^{-\frac{3N}{\rho u}} du + O(\rho^{-2}) \\ &= \int_{\frac{3N}{\rho\delta}}^\infty \left(1 - \frac{9N}{\rho^2 t}\right) \frac{3N}{\rho^3 t} \frac{3N}{\rho t^2} e^{-t} dt + O(\rho^{-2}) \\ &= \underbrace{\int_{\frac{3N}{\rho\delta}}^\infty \frac{9N^2}{\rho^4 t^3} e^{-t} dt}_{A_{11}} - \underbrace{\int_{\frac{3N}{\rho\delta}}^\infty \frac{81N^3}{\rho^6 t^4} e^{-t} dt}_{A_{12}} + O(\rho^{-2}) \end{aligned} \tag{F.33}$$

Lemma 2: Integration by Parts

$$\int u\, dv = uv - \int v\, du$$

Especially,

$$\begin{aligned}\int \frac{e^{-t}}{t^n}dt &= \int e^{-t}dt^{-(n-1)} \\ &= -\frac{1}{n-1}\left(\frac{e^{-t}}{t^{n-1}} - \int t^{-(n-1)}de^{-t}\right) \\ &= -\frac{1}{n-1}\left(\frac{e^{-t}}{t^{n-1}} + \int t^{-(n-1)}e^{-t}d\right) \\ &= O\left(\left(t^{-1}\right)^{n-1}\right) \end{aligned} \tag{F.34}$$

Applying Lemma 2 on A_{11} where $n = 3$ in Eq. (F.34), we obtain

$$\begin{aligned}A_{11} &= \frac{9N^2}{\rho^4}\int_{\frac{3N}{\rho\delta}}^{\infty} t^{-3}e^{-t}dt \\ &= \frac{9N^2}{\rho^4}O\left(\left(\frac{3N}{\rho\delta}^{-1}\right)^2\right) \\ &= O(\rho^{-2})\end{aligned}$$

Similarly, applying Lemma 2 on A_{12} where $n = 4$

$$\begin{aligned}A_{12} &= \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{81N^3}{\rho^6 t^4}e^{-t}dt \\ &= \frac{81N^3}{\rho^6}O\left(\left(\frac{3N}{\rho\delta}^{-1}\right)^3\right) \\ &= O(\rho^{-3})\end{aligned}$$

Above all, substitute A_{11} and A_{12} back to A_1 in Eq.

$$A_1 = O(\rho^{-2}) \tag{F.35}$$

Applying Lemma 1 on A_2, we have

$$\begin{aligned} A_2 &= \int_{\delta}^{\infty} \left(1 - \frac{3N}{\rho u} + O(\rho^{-2})\right) \frac{u}{\rho^2} e^{-\frac{3u}{\rho}} du \\ &= \int_{\delta}^{\infty} \left(1 - \frac{3N}{\rho u}\right) \frac{u}{\rho^2} e^{-\frac{3u}{\rho}} du + O(\rho^{-2}) \end{aligned}$$

Replace u with $t = \frac{3u}{\rho}$, so that $u = \frac{\rho t}{3}$ and $du = \frac{\rho}{3} dt$, A_2 changes into

$$\begin{aligned} A_2 &= \int_{\delta}^{\infty} \left(1 - \frac{3N}{\rho u}\right) \frac{u}{\rho^2} e^{-\frac{3u}{\rho}} du + O(\rho^{-2}) \\ &= \underbrace{\int_{\frac{3\delta}{\rho}}^{\infty} \frac{t}{9} e^{-t} dt}_{A_{21}} - \underbrace{\int_{\frac{3\delta}{\rho}}^{\infty} \frac{9N}{\rho^2 t} \frac{t}{9} e^{-t} dt}_{A_{22}} + O(\rho^{-2}) \end{aligned} \tag{F.36}$$

Lemma 3 Applying integration by parts,

$$\begin{aligned} \int t^n e^{-t} dt &= -\int t^n de^{-t} \\ &= -\left(t^n e^{-t} - \int e^{-t} dt^n\right) \\ &= -\left(t^n e^{-t} - \int n t^{n-1} e^{-t} d\right) \\ &= O\left(t^n\right) \end{aligned} \tag{F.37}$$

Applying Lemma 3 on A_{21} where $n = 1$ in Eq. (F.37),

$$\begin{aligned}
A_{21} &= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{t}{9} e^{-t} dt \\
&= -\frac{1}{9}\left[te^{-t}\Big|_{\frac{3\delta}{\rho}}^{\infty} - \int_{\frac{3\delta}{\rho}}^{\infty} e^{-t} dt \right] \\
&= -\frac{1}{9}\left[0 - \frac{3\delta}{\rho} e^{-\frac{3\delta}{\rho}} + e^{-t}\Big|_{\frac{3\delta}{\rho}}^{\infty} \right] \\
&= -\frac{1}{9}\left[-\frac{3\delta}{\rho} e^{-\frac{3\delta}{\rho}} - e^{-\frac{3\delta}{\rho}} \right] \\
&= \left[\frac{\delta}{3\rho} + \frac{1}{9} \right] e^{-\frac{3\delta}{\rho}}
\end{aligned}$$

Applying Lemma 1,

$$\begin{aligned}
A_{21} &= \left[\frac{\delta}{3\rho} + \frac{1}{9} \right] \left[1 - \frac{3\delta}{\rho} + O(\rho^{-2}) \right] \\
&= \left[\frac{\delta}{3\rho} + \frac{1}{9} \right] \left[1 - \frac{3\delta}{\rho} \right] + O(\rho^{-2}) \\
&= \frac{\delta}{3\rho} + \frac{1}{9} - \frac{1}{9}\frac{3\delta}{\rho} + O(\rho^{-2}) \\
&= \frac{1}{9} + O(\rho^{-2})
\end{aligned}$$

Similarly,

$$\begin{aligned}
A_{22} &= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{N}{\rho^2} e^{-t} dt \\
&= -\frac{N}{\rho^2} e^{-t}\Big|_{\frac{3\delta}{\rho}}^{\infty} \\
&= \frac{N}{\rho^2} e^{-\frac{3\delta}{\rho}}
\end{aligned}$$

Using Lemma 1, A_{22} becomes

$$\begin{aligned} A_{22} &= \frac{N}{\rho^2}\left[1 - \frac{3\delta}{\rho} + O(\rho^{-2})\right] \\ &= O(\rho^{-2}) \end{aligned}$$

Substituting A_{21} and A_{22} back into A_2 in Eq. (F.36), we have

$$A_2 = \frac{1}{9} + O(\rho^{-2}) \tag{F.38}$$

Above all, substituting Eqs. (F.35) and (F.38) into Eq. (F.32),

$$\begin{aligned} A &= \left(1 - \frac{3M}{\rho}\right)\frac{9}{2}\text{erfc}(\frac{\sqrt{6}}{2})\left[\frac{1}{9} + O(\rho^{-2})\right] + O(\rho^{-2}) \\ &= \frac{9}{2}\text{erfc}(\frac{\sqrt{6}}{2})\left[\frac{1}{9} - \frac{M}{3\rho} + O(\rho^{-2})\right] + O(\rho^{-2}) \\ &= \frac{1}{2}\text{erfc}(\frac{\sqrt{6}}{2}) - \frac{3M}{2}\text{erfc}(\frac{\sqrt{6}}{2})\rho^{-1} + O(\rho^{-2}) \end{aligned} \tag{F.39}$$

b) **Integral** B:

$$\begin{aligned} B &= \int_0^\infty \frac{u}{\rho^2}\sqrt{\frac{27}{2\pi}}a^{-2}e^{-\frac{3}{2}a}du \\ &= \underbrace{\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}}_{CONST}\int_0^\infty \frac{u}{\rho^2}a^{-2}e^{-\frac{3}{2}(a-1)}du \\ &= \underbrace{\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}}_{CONST}\int_0^\infty \frac{u}{\rho^2}\frac{\rho^2u^2}{(2u^2 + (2M+\rho)u + 2N)^2}e^{-\frac{3u}{\rho}}e^{-\frac{3N}{\rho u}}e^{-\frac{3M}{\rho}}du \end{aligned}$$

where $a = \frac{2u^2+(2M+\rho)u+2N}{\rho u}$.

Applying Lemma 1 on B and divide B into two parts B_1 and B_2, it becomes

$$B = \underbrace{\sqrt{\frac{27}{2\pi}} e^{-\frac{3}{2}} \left[1 - \frac{3M}{\rho}\right]}_{CONST} \left\{ \left[\underbrace{\int_0^\delta \left[1 - \frac{3u}{\rho}\right] \frac{u}{\rho^2} \frac{\rho^2 u^2}{(2u^2+(2M+\rho)u+2N)^2} e^{-\frac{3N}{\rho u}} du}_{B_1} \right] \right.$$
$$\left. + \left[\underbrace{\int_\delta^\infty \left[1 - \frac{3N}{\rho u}\right] \frac{u}{\rho^2} \frac{\rho^2 u^2}{(2u^2+(2M+\rho)u+2N)^2} e^{-\frac{3u}{\rho}} du}_{B_2} \right] \right\} + O(\rho^{-2}) \tag{F.40}$$

Divide B_1 into two parts B_{11} and B_{12},

$$B_1 = \int_0^\delta \left(1 - \frac{3u}{\rho}\right) \frac{u}{\rho^2} \frac{\rho^2 u^2}{(2u^2+(2M+\rho)u+2N)^2} e^{-\frac{3N}{\rho u}} du \tag{F.41}$$
$$= \underbrace{\int_0^\delta \frac{u^3}{(2u^2+(2M+\rho)u+2N)^2} e^{-\frac{3N}{\rho u}} du}_{B_{11}} - \underbrace{\int_0^\delta \frac{3u^4}{\rho(2u^2+(2M+\rho)u+2N)^2} e^{-\frac{3N}{\rho u}} du}_{B_{12}} \tag{F.42}$$

Replace u with $t = \frac{3N}{\rho u}$ in B_{11} and B_{12}, so that $u = \frac{3N}{\rho t}$ and $du = \frac{3N}{\rho}(-1/t^2)dt$. So

B_{11} changes into

$$\begin{aligned}
B_{11} &= \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{\frac{27N^3}{\rho^3 t^3}}{(\frac{18N^2}{\rho^2 t^2} + (2M+\rho)\frac{3N}{\rho t} + 2N)^2} e^{-t} \frac{3N}{\rho}(1/t^2)dt \\
&= 81N^2 \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{\frac{1}{\rho^4 t^5}}{(\frac{18N}{\rho^2 t^2} + \frac{6M}{\rho t} + \frac{3}{t} + 2)^2} e^{-t} dt \\
&= 81N^2 \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{\frac{1}{t}}{(18N + 6M\rho t + 3\rho^2 t + 2\rho^2 t^2)^2} e^{-t} dt \\
&= 81N^2 \int_{\frac{3N}{\rho\delta}}^{\infty} O(\rho^{-4}) t^{-1} e^{-t} dt \\
&= O(\rho^{-4})
\end{aligned}$$

Similarly,

$$\begin{aligned}
B_{12} &= \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{3\frac{81N^4}{\rho^4 t^4}}{\rho(\frac{18N^2}{\rho^2 t^2} + (2M+\rho)\frac{3N}{\rho t} + 2N)^2} e^{-t} \frac{3N}{\rho}(1/t^2)dt \\
&= 9 \times 81N^3 \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{\frac{1}{\rho^5 t^6}}{(\frac{18N}{\rho^2 t^2} + \frac{6M}{\rho t} + \frac{3}{t} + 2)^2} e^{-t} dt \\
&= 9 \times 81N^3 \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{\frac{1}{\rho t^2}}{(18N + 6M\rho t + 3\rho^2 t + 2\rho^2 t^2)^2} e^{-t} dt \\
&= 9 \times 81N^3 \int_{\frac{3N}{\rho\delta}}^{\infty} O(\rho^{-2}) t^{-2} e^{-t} dt \\
&= O(\rho^{-2})
\end{aligned}$$

Substituting B_{11} and B_{12} back into B_1 in Eq. (F.41), we obtain

$$B_1 = O(\rho^{-2}) \tag{F.43}$$

We also divide B_2 into two parts,

$$
\begin{aligned}
B_2 &= \int_{\delta}^{\infty} \left(1 - \frac{3N}{\rho u}\right) \frac{u}{\rho^2} \frac{\rho^2 u^2}{(2u^2 + (2M+\rho)u + 2N)^2} e^{-\frac{3u}{\rho}} du \\
&= \underbrace{\int_{\delta}^{\infty} \frac{u^3}{(2u^2 + (2M+\rho)u + 2N)^2} e^{-\frac{3u}{\rho}} du}_{B_{21}} - \underbrace{\int_{\delta}^{\infty} \frac{3Nu^2}{\rho(2u^2 + (2M+\rho)u + 2N)^2} e^{-\frac{3u}{\rho}} du}_{B_{22}}
\end{aligned}
\tag{F.44}
$$

Replace u with $t = \frac{3u}{\rho}$ in B_{21} and B_{22}, so that $u = \frac{\rho t}{3}$ and $du = \frac{\rho}{3} dt$, B_{21} changes into

$$
\begin{aligned}
B_{21} &= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{\left(\frac{\rho t}{3}\right)^3}{(2\left(\frac{\rho t}{3}\right)^2 + (2M+\rho)\left(\frac{\rho t}{3}\right) + 2N)^2} \frac{\rho}{3} e^{-t} dt \\
&= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{\left(\frac{\rho^4 t^3}{81}\right)}{(\frac{2\rho^2 t^2}{9} + \frac{\rho^2 t}{3} + \frac{2Mt\rho}{3} + 2N)^2} e^{-t} dt \\
&= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{\rho^4 t^3}{(2\rho^2 t^2 + 3\rho^2 t + 6Mt\rho + 18N)^2} e^{-t} dt
\end{aligned}
\tag{F.45}
$$

Lemma 4

$$
\begin{aligned}
&\frac{\rho^m}{a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_0\rho^0} \\
=&\frac{\rho^m}{a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_0\rho^0} - \frac{\rho^m}{a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_q\rho^q} \\
&+ \frac{\rho^m}{a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_q\rho^q} \\
=&\frac{\rho^m(a_{q-1}\rho^{q-1} + \cdots + a_0\rho^0)}{O(\rho^{2n})} + \frac{\rho^m}{a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_q\rho^q} \\
=&O\left(\rho^{m+q-1-2n}\right) + \frac{\rho^m}{a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_q\rho^q}
\end{aligned}
\tag{F.46}
$$

Applying Lemma 4 on the long polynomial within the integral in Eq. (F.45). According to Eq. (F.46), we have $m = n = 4$, and we are intended to have $O(\rho^{-2})$ term, which means $m + q - 1 - 2n = -2$, therefore, $q = 3$.

$$\begin{aligned}
&\frac{\rho^4 t^3}{(2\rho^2 t^2 + 3\rho^2 t + 6Mt\rho + 18N)^2} \\
&= \frac{\rho^4 t^3}{4\rho^4 t^4 + 4\rho^2(3\rho^2 + 6M\rho)t^3 + [(3\rho^2 + 6M\rho)^2 + 72N\rho^2]t^2 + 36(3\rho^2 + 6M\rho)t + 18^2N^2} \\
=&\frac{\rho^4 t^3}{4\rho^4 t^4 + 4\rho^2(3\rho^2 + 6M\rho)t^3 + 9\rho^4 t^2 + 36M\rho^3 t^2} + \frac{\rho^4 O(\rho^2)}{O(\rho^8)} \\
=&\frac{\rho t}{4\rho t^2 + 4(3\rho + 6M)t + 9\rho + 36M} + O(\rho^{-2})
\end{aligned} \tag{F.47}$$

Simplify Eq. (F.47) with partial fraction, it becomes

$$\begin{aligned}
&\frac{t}{4\left(t^2 + \frac{(3\rho+6M)t}{\rho} + \frac{9\rho+36M}{4\rho}\right)} + O(\rho^{-2}) \\
=&\frac{t}{4\left[\left(t + \frac{(3\rho+6M)}{2\rho}\right)^2 - \left(\frac{3M}{\rho}\right)^2\right]} + O(\rho^{-2}) \\
=&\frac{t}{4\left(t + \frac{(3\rho+12M)}{2\rho}\right)\left(t + \frac{3}{2}\right)} + O(\rho^{-2}) \\
=&\frac{\rho}{24M}\left[\frac{t}{t + \frac{3}{2}} - \frac{t}{t + \frac{(3\rho+12M)}{2\rho}}\right] + O(\rho^{-2}) \\
=&\frac{\rho}{24M}\left[-\frac{\frac{3}{2}}{t + \frac{3}{2}} + \frac{\frac{(3\rho+12M)}{2\rho}}{t + \frac{(3\rho+12M)}{2\rho}}\right] + O(\rho^{-2})
\end{aligned} \tag{F.48}$$

Lemma 5: Binomial Series

In mathematics, the binomial series is the Taylor series at $x = 0$ of the function f

given by $f(x) = (1+x)^{\alpha}$, where $\alpha \in \mathbf{C}$ is an arbitrary complex number. Explicitly,

$$\begin{aligned}(1+x)^{\alpha} &= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots, \end{aligned} \tag{F.49}$$

Therefore,

$$\begin{aligned}\frac{1}{a_0 + \frac{a_1}{\rho}} &= \frac{1}{a_0(1+\frac{a_1}{a_0\rho})} \\ &= \frac{1}{a_0}\left(1 - \frac{a_1}{a_0\rho} + O(\rho^{-2})\right)\end{aligned} \tag{F.50}$$

Lemma 6: Taylor Series of Natural Logarithm

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \text{for} \quad |x| \le 1 \tag{F.51}$$

Lemma 7

$$\begin{aligned}&\int_{\frac{3\delta}{\rho}}^{\infty} \frac{1}{t+3/2} e^{-t} dt \\ &= \int_0^{\infty} \frac{1}{t+3/2} e^{-t} dt - \int_0^{\frac{3\delta}{\rho}} \frac{1}{t+3/2} e^{-t} dt \\ &= \int_0^{\infty} \frac{1}{t+3/2} e^{-t} dt - \left(-\frac{3\delta}{\rho} + \frac{5}{2} ln(t + \frac{3}{2})\Big|_0^{\frac{3\delta}{\rho}}\right) \\ &= \int_0^{\infty} \frac{1}{t+3/2} e^{-t} dt - \left(-\frac{3\delta}{\rho} + \frac{5}{2}\ln(1 + \frac{2\delta}{\rho})\right)\end{aligned} \tag{F.52}$$

Applying Lemma 6 on natural logarithm, we obtain

$$\ln(1+\frac{2\delta}{\rho}) = 1 + \frac{2\delta}{\rho} + O(\rho^{-2})$$

when $\rho \to \infty$.

Therefore, Eq. (F.52) becomes

$$\begin{aligned} &\int_0^\infty \frac{1}{t+3/2}e^{-t}dt - \left(-\frac{3\delta}{\rho} + \frac{5\delta}{\rho}\right) \\ =&\int_0^\infty \frac{1}{t+3/2}e^{-t}dt - \frac{2\delta}{\rho} \end{aligned}$$

Substituting Eq. (F.48) into Eq. (F.45), and applying Eq. (F.50) in Lemma 5, we obtain

$$\begin{aligned} B_{21} &= \frac{\rho}{24M}\left\{\int_{\frac{3\delta}{\rho}}^\infty \frac{-\frac{3}{2}}{t+3/2}e^{-t}dt + \int_{\frac{3\delta}{\rho}}^\infty \frac{(3\rho+12M)}{2\rho}\left[\frac{-6M/\rho}{(t+3/2)^2} + \frac{1}{t+3/2}\right]e^{-t}dt\right\} \\ &= \frac{\rho}{24M}\left\{\int_{\frac{3\delta}{\rho}}^\infty \frac{\frac{6M}{\rho}}{t+3/2}e^{-t}dt + \int_{\frac{3\delta}{\rho}}^\infty \frac{(3\rho+12M)}{2\rho}\frac{-6M/\rho}{(t+3/2)^2}e^{-t}dt\right\} \\ &= \frac{1}{4}\left\{\int_{\frac{3\delta}{\rho}}^\infty \frac{1}{t+3/2}e^{-t}dt - \int_{\frac{3\delta}{\rho}}^\infty \frac{(3\rho+12M)}{2\rho}\frac{1}{(t+3/2)^2}e^{-t}dt\right\} \end{aligned}$$

Applying Lemma 2, in Eq. (F.34), $n=2$, therefore, B_{21} becomes

$$B_{21} = \frac{1}{4}\left\{\int_{\frac{3\delta}{\rho}}^\infty \frac{1}{t+3/2}e^{-t}dt + \frac{(3\rho+12M)}{2\rho}\left[-\frac{e^{-\frac{3\delta}{\rho}}}{\frac{3\delta}{\rho}+\frac{3}{2}} + \int_{\frac{3\delta}{\rho}}^\infty \frac{1}{t+3/2}e^{-t}dt\right]\right\}$$

Applying Lemma 1,

$$e^{-\frac{3\delta}{\rho}} = 1 - \frac{3\delta}{\rho} + O(\rho^{-2}) \tag{F.53}$$

Applying Eq. (F.50) in Lemma 5,

$$\frac{1}{\frac{3\delta}{\rho} + \frac{3}{2}} = \frac{2}{3}\left(1 - \frac{3\delta}{\rho}/\frac{3}{2} + O(\rho^{-2})\right) \tag{F.54}$$

$$= \frac{2}{3}\left(1 - \frac{2\delta}{\rho}\right) + O(\rho^{-2}) \tag{F.55}$$

Substituting Eqs. (F.53) and (F.62) into B_{21}, it becomes

$$B_{21} = \frac{1}{4}\left\{\int_{\frac{3\delta}{\rho}}^{\infty} \frac{1}{t+3/2} e^{-t} dt + \frac{(3\rho + 12M)}{2\rho}\left[-\frac{2}{3}\left(1 - \frac{2\delta}{\rho}\right)\left(1 - \frac{3\delta}{\rho}\right) + \int_{\frac{3\delta}{\rho}}^{\infty} \frac{1}{t+3/2} e^{-t} dt\right]\right\}$$
$$= -\frac{1}{4} + \frac{5\delta}{4\rho} - \frac{M}{\rho} + \left(\frac{5}{8} + \frac{3M}{2\rho}\right)\int_{\frac{3\delta}{\rho}}^{\infty} \frac{1}{t+3/2} e^{-t} dt$$

Substituting Lemma 7 into B_{21}, we have

$$B_{21} = -\frac{1}{4} + \frac{5\delta}{4\rho} - \frac{M}{\rho} + \left(\frac{5}{8} + \frac{3M}{2\rho}\right)\left(\int_{0}^{\infty} \frac{1}{t+3/2} e^{-t} dt - \frac{2}{\rho}\right)$$
$$= -\frac{1}{4} - \frac{M}{\rho} + \left(\frac{5}{8} + \frac{3M}{2\rho}\right)\int_{0}^{\infty} \frac{1}{t+3/2} e^{-t} dt \tag{F.56}$$

$$
\begin{aligned}
B_{22} &= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{3N\left(\frac{\rho t}{3}\right)^2}{\rho(2\left(\frac{\rho t}{3}\right)^2 + (2M+\rho)\left(\frac{\rho t}{3}\right) + 2N)^2} \frac{\rho}{3} e^{-t} dt \\
&= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{N\left(\frac{\rho^2 t^2}{9}\right)}{(\frac{2\rho^2 t^2}{9} + \frac{\rho^2 t}{3} + \frac{2Mt\rho}{3} + 2N)^2} e^{-t} dt \\
&= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{9N\rho^2 t^2}{(2\rho^2 t^2 + 3\rho^2 t + 6Mt\rho + 18N)^2} e^{-t} dt \\
&= \int_{\frac{3\delta}{\rho}}^{\infty} O(\rho^{-2}) e^{-t} dt \\
&= O(\rho^{-2}) \qquad \text{(F.57)}
\end{aligned}
$$

Substituting Eqs. (F.56) and (F.57) back into B_2 in Eq. (F.44), we obtain

$$
B_2 = -\frac{1}{4} - \frac{M}{\rho} + \left(\frac{5}{8} + \frac{3M}{2\rho}\right) \int_0^{\infty} \frac{1}{t+3/2} e^{-t} dt + O(\rho^{-2}) \qquad \text{(F.58)}
$$

Above all, Eqs. (F.43) and (F.58) back into B in Eq. (F.40), we obtain

$$
\begin{aligned}
B &= \sqrt{\frac{27}{2\pi}} e^{-\frac{3}{2}} \left[1 - \frac{3M}{\rho}\right] \left\{-\frac{1}{4} - \frac{M}{\rho} + \left(\frac{5}{8} + \frac{3M}{2\rho}\right) \int_0^{\infty} \frac{1}{t+3/2} e^{-t} dt + O(\rho^{-2})\right\} + O(\rho^{-2}) \\
&= \sqrt{\frac{27}{2\pi}} e^{-\frac{3}{2}} \left[-\frac{1}{4} + \frac{5}{8} \int_0^{\infty} \frac{1}{t+3/2} e^{-t} dt + \left(-\frac{M}{4} - \frac{3M}{8} \int_0^{\infty} \frac{1}{t+3/2} t^{-2} e^{-t} dt\right) \rho^{-1}\right] \\
&\quad + O(\rho^{-2}) \qquad \text{(F.59)}
\end{aligned}
$$

c) **Integral C**:

Similarly with B,

$$\begin{aligned}
C &= \int_0^\infty \frac{u}{\rho^2}\sqrt{\frac{27}{2\pi}}a^{-1}e^{-\frac{3}{2}a}du \\
&= \underbrace{\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}}_{CONST}\int_0^\infty \frac{u}{\rho^2}a^{-1}e^{-\frac{3}{2}(a-1)}du \\
&= \underbrace{\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}}_{CONST}\int_0^\infty \frac{u}{\rho^2}a^{-1}e^{-\frac{3}{2}(a-1)}du \\
&= \underbrace{\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}}_{CONST}\int_0^\infty \frac{u}{\rho^2}\frac{\rho u}{(2u^2+(2M+\rho)u+2N)}e^{-\frac{3u}{\rho}}e^{-\frac{3N}{\rho u}}e^{-\frac{3M}{\rho}}du
\end{aligned}$$

Applying Lemma 1 on C, it becomes

$$\begin{aligned}
&\underbrace{\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}\left[1-\frac{3M}{\rho}\right]}_{CONST}\left\{\left[\underbrace{\int_0^\delta\left(1-\frac{3u}{\rho}\right)\frac{u}{\rho^2}\frac{\rho u}{(2u^2+(2M+\rho)u+2N)}e^{-\frac{3N}{\rho u}}du}_{C_1}\right]\right. \\
&\left.+\left[\underbrace{\int_\delta^\infty\left(1-\frac{3N}{\rho u}\right)\frac{u}{\rho^2}\frac{\rho u}{(2u^2+(2M+\rho)u+2N)}e^{-\frac{3u}{\rho}}du}_{C_2}\right]\right\} \qquad \text{(F.60)}
\end{aligned}$$

Divide C_1 into two parts,

$$\begin{aligned}
C_1 &= \int_0^\delta\left(1-\frac{3u}{\rho}\right)\frac{u}{\rho^2}\frac{\rho u}{(2u^2+(2M+\rho)u+2N)}e^{-\frac{3N}{\rho u}}du \\
&= \underbrace{\int_0^\delta \frac{u^2}{\rho(2u^2+(2M+\rho)u+2N)}e^{-\frac{3N}{\rho u}}du}_{C_{11}} - \underbrace{\int_0^\delta \frac{3u^3}{\rho^2(2u^2+(2M+\rho)u+2N)}e^{-\frac{3N}{\rho u}}du}_{C_{12}}
\end{aligned}$$

Replace u with $t=\frac{3N}{\rho u}$ in C_{11} and C_{12}, so that $u=\frac{3N}{\rho t}$ and $du=\frac{3N}{\rho}(-1/t^2)dt$. So

C_{11} changes into

$$\begin{aligned}
C_{11} &= \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{\frac{9N^2}{\rho^3 t^2}}{(\frac{18N^2}{\rho^2 t^2} + (2M+\rho)\frac{3N}{\rho t} + 2N)} e^{-t} \frac{3N}{\rho}(1/t^2)dt \\
&= 27N^2 \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{\frac{1}{\rho^4 t^4}}{(\frac{18N}{\rho^2 t^2} + \frac{6M}{\rho t} + \frac{3}{t} + 2)} e^{-t} dt \\
&= 27N^2 \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{1}{18N\rho^2 t^2 + 6M\rho^3 t^3 + 3\rho^4 t^3 + 2\rho^4 t^4} e^{-t} dt \\
&= 27N^2 \int_{\frac{3N}{\rho\delta}}^{\infty} t^{-2} O(\rho^{-2}) e^{-t} dt \\
&= O(\rho^{-2})
\end{aligned}$$

In the same way,

$$\begin{aligned}
C_{12} &= \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{\frac{81N^3}{\rho^5 t^3}}{(\frac{18N^2}{\rho^2 t^2} + (2M+\rho)\frac{3N}{\rho t} + 2N)} e^{-t} \frac{3N}{\rho}(1/t^2)dt \\
&= 81N^3 \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{\frac{1}{\rho^6 t^5}}{(\frac{18N}{\rho^2 t^2} + \frac{6M}{\rho t} + \frac{3}{t} + 2)} e^{-t} dt \\
&= 81N^3 \int_{\frac{3N}{\rho\delta}}^{\infty} \frac{1}{18N\rho^4 t^3 + 6M\rho^5 t^4 + 3\rho^6 t^4 + 2\rho^6 t^5} e^{-t} dt \\
&= 81N^2 \int_{\frac{3N}{\rho\delta}}^{\infty} O(\rho^{-4}) t^{-3} e^{-t} dt \\
&= O(\rho^{-4})
\end{aligned}$$

Therefore, we can acquire that

$$C_1 = O(\rho^{-2}) \tag{F.61}$$

Divide C_2 into two parts,

$$\begin{aligned}
C_2 &= \int_{\delta}^{\infty} \left(1 - \frac{3N}{\rho u}\right) \frac{u}{\rho^2} \frac{\rho u}{(2u^2 + (2M+\rho)u + 2N)} e^{-\frac{3u}{\rho}} du \\
&= \underbrace{\int_{\delta}^{\infty} \frac{u^2}{\rho(2u^2 + (2M+\rho)u + 2N)} e^{-\frac{3u}{\rho}} du}_{C_{21}} - \underbrace{\int_{\delta}^{\infty} \frac{3Nu}{\rho^2(2u^2 + (2M+\rho)u + 2N)} e^{-\frac{3u}{\rho}} du}_{C_{21}}
\end{aligned} \tag{F.62}$$

Replace u with $t = \frac{3u}{\rho}$ in C_{21} and C_{22}, so that $u = \frac{\rho t}{3}$ and $du = \frac{\rho}{3} dt$, C_{21} changes into

$$\begin{aligned}
C_{21} &= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{\left(\frac{\rho t}{3}\right)^2}{\rho(2\left(\frac{\rho t}{3}\right)^2 + (2M+\rho)\left(\frac{\rho t}{3}\right) + 2N)} \frac{\rho}{3} e^{-t} dt \\
&= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{\frac{\rho^2 t^2}{27}}{\frac{2\rho^2 t^2}{9} + \frac{\rho^2 t}{3} + \frac{2Mt\rho}{3} + 2N} e^{-t} dt \\
&= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{\rho^2 t^2}{(6t^2 + 9t)\rho^2 + 18Mt\rho + 54N} e^{-t} dt
\end{aligned}$$

Applying Lemma 4 on the long polynomial within the integral in Eq. (F.45). According to Eq. (F.46), we have $m = n = 2$, and we are intended to have $O(\rho^{-2})$

term, which means $m + q - 1 - 2n = -2$, therefore, $q = 1$.

$$\begin{aligned}
C_{21} &= \int_{\frac{3\delta}{\rho}}^{\infty} \left[\frac{\rho^2 t^2}{6\rho^2 t^2 + 9\rho^2 t + 18Mt\rho} + O(\rho^{-2}) \right] e^{-t} dt \\
&= \int_{\frac{3\delta}{\rho}}^{\infty} \frac{\rho t}{6\rho t + 9\rho + 18M} e^{-t} dt + O(\rho^{-2}) \\
&= \frac{1}{6} \int_{\frac{3\delta}{\rho}}^{\infty} \frac{t}{t + \frac{3\rho + 6M}{2\rho}} e^{-t} dt + O(\rho^{-2}) \\
&= \frac{1}{6} \int_{\frac{3\delta}{\rho}}^{\infty} \left[1 - \frac{\frac{3\rho+6M}{2\rho}}{t + \frac{3\rho+6M}{2\rho}} \right] e^{-t} dt + O(\rho^{-2})
\end{aligned}$$

Applying Lemma 5 on C_{21},

$$\begin{aligned}
C_{21} &= \frac{1}{6} \left[e^{-(\frac{3\delta}{\rho})} - \frac{3\rho + 6M}{2\rho} \int_{\frac{3\delta}{\rho}}^{\infty} \left(\frac{-3M}{\rho} (t + 3/2)^{-2} + \frac{1}{t + 3/2} \right) e^{-t} dt \right] + O(\rho^{-2}) \\
&= \frac{1}{6} \left[1 - \frac{3\delta}{\rho} - \left(\frac{3}{2} + \frac{3M}{\rho} \right) \left[-\frac{3M}{\rho} \frac{e^{-\frac{3\delta}{\rho}}}{\frac{3\delta}{2} + \frac{3}{\rho}} + \left(\frac{3M}{\rho} + 1 \right) \int_{\frac{3\delta}{\rho}}^{\infty} \frac{1}{t + 3/2} e^{-t} dt \right] \right] + O(\rho^{-2})
\end{aligned}$$

Substituting Eqs. (F.53) and (F.62) into C_{21}, it becomes

$$\begin{aligned}
C_{21} &= \frac{1}{6} \left[1 - \frac{3\delta}{\rho} - \left(\frac{3}{2} + \frac{3M}{\rho} \right) \left[-\frac{3M}{\rho} \left(-\frac{2}{3} \right) \left(1 - \frac{2\delta}{\rho} \right) \left(1 - \frac{3\delta}{\rho} \right) \right.\right. \\
&\quad + \left.\left. \left(\frac{3M}{\rho} + 1 \right) \int_{\frac{3\delta}{\rho}}^{\infty} \frac{1}{t + 3/2} e^{-t} dt \right] \right] \\
&= \frac{1}{6} \left[1 - \frac{3\delta}{\rho} + \frac{3M}{\rho} - \left(\frac{3}{2} + \frac{15M}{2\rho} \right) \left(\int_{\frac{3\delta}{\rho}}^{\infty} \frac{1}{t + 3/2} e^{-t} dt \right) \right] + O(\rho^{-2})
\end{aligned}$$

Substituting Lemma 7 into C_{21},

$$
\begin{aligned}
C_{21} &= \frac{1}{6}\left[1 - \frac{3\delta}{\rho} + \frac{3M}{\rho} - \left(\frac{3}{2} + \frac{15M}{2\rho}\right)\left(\int_0^\infty \frac{1}{t+3/2}e^{-t}dt - \frac{2\delta}{\rho}\right)\right] + O(\rho^{-2}) \\
&= \frac{1}{6} - \frac{1}{4}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt + \frac{M}{2\rho} - \frac{5M}{4\rho}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt + O(\rho^{-2})
\end{aligned}
$$

$$
\begin{aligned}
C_{22} &= \int_{\frac{3\delta}{\rho}}^\infty \frac{3N\left(\frac{\rho t}{3}\right)}{\rho^2(2\left(\frac{\rho t}{3}\right)^2 + (2M+\rho)\left(\frac{\rho t}{3}\right) + 2N)}\frac{\rho}{3}e^{-t}dt \\
&= \int_{\frac{3\delta}{\rho}}^\infty \frac{\frac{Nt}{3}}{\frac{2\rho^2 t^2}{9} + \frac{\rho^2 t}{3} + \frac{2Mt\rho}{3} + 2N}e^{-t}dt \\
&= \int_{\frac{3\delta}{\rho}}^\infty \frac{3Nt}{(2t^2+3t)\rho^2 + 6Mt\rho + 18N}e^{-t}dt \\
&= \int_{\frac{3\delta}{\rho}}^\infty O(\rho^{-2})t^{-1}e^{-t}dt \\
&= O(\rho^{-2})
\end{aligned}
$$

Therefore, substituting C_{21} and C_{22} back into C_2 in Eq. (F.62), we obtain

$$
C_2 = \frac{1}{6} - \frac{1}{4}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt + \frac{M}{2\rho} - \frac{5M}{4\rho}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt + O(\rho^{-2}) \quad \text{(F.63)}
$$

Above all, substituting Eqs. (F.61) and (F.63) back into C_2 in Eq. (F.60), we obtain

$$\begin{aligned} C &= \sqrt{\frac{27}{2\pi}} e^{-\frac{3}{2}} \left(1 - \frac{3M}{\rho}\right) \left[\frac{1}{6} - \frac{1}{4}\int_0^\infty \frac{1}{t+3/2} e^{-t} dt + \frac{M}{2\rho} - \frac{5M}{4\rho}\int_0^\infty \frac{1}{t+3/2} e^{-t} dt + O(\rho^{-2})\right] \\ &= \sqrt{\frac{27}{2\pi}} e^{-\frac{3}{2}} \left[\frac{1}{6} - \frac{1}{4}\int_0^\infty \frac{1}{t+3/2} e^{-t} dt - \frac{5M}{4}\int_0^\infty \frac{1}{t+3/2} e^{-t} dt \rho^{-1} + O(\rho^{-2})\right] \end{aligned} \tag{F.64}$$

d) **Integral** D:

$$D = \int_0^\infty \frac{u}{\rho^2} \frac{3}{2} a^{-\frac{5}{2}} \mathrm{erfc}(\frac{\sqrt{6a}}{2}) du \tag{F.65}$$

First we convert error function into Q function,

$$\mathrm{Q}(x) = \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta$$

Nextly we express Q function in trigonometric form,

$$\begin{aligned} \mathrm{erfc}(\frac{\sqrt{6a}}{2}) &= 2\mathrm{Q}(\sqrt{3a}) \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{3a}{2\sin^2\theta}} d\theta \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{3a(1+\cot^2\theta)}{2}} d\theta \end{aligned} \tag{F.66}$$

If we replace $\cot\theta$ with b, we have $db = -(1+\cot^2\theta)d\theta$, Eq. (F.66) becomes

$$\operatorname{erfc}(\frac{\sqrt{6a}}{2}) = \frac{2}{\pi}\int_0^\infty \frac{e^{-\frac{3a(1+b^2)}{2}}}{1+b^2}db \tag{F.67}$$

Therefore, substituting Eq. (F.67) into Eq. (F.68),

$$\begin{aligned} D &= \int_0^\infty \frac{u}{\rho^2}\frac{3}{2}a^{-\frac{5}{2}}\frac{2}{\pi}\int_0^\infty \frac{e^{-\frac{3a(1+b^2)}{2}}}{1+b^2}dbdu \\ &= \frac{3}{\pi}\int_0^\infty \frac{1}{1+b^2}\left[\int_0^\infty \frac{u}{\rho^2}a^{-\frac{5}{2}}e^{-\frac{3a(1+b^2)}{2}}du\right]db \\ &= \frac{3}{\pi}\int_0^\infty \frac{1}{1+b^2}e^{-\frac{3(1+b^2)}{2}}\left[\underbrace{\int_0^\infty \frac{u}{\rho^2}a^{-\frac{5}{2}}e^{-\frac{3(a-1)(1+b^2)}{2}}du}_{D_0}\right]db \end{aligned} \tag{F.68}$$

Applying Lemma 1 on D_0 and defining $c = 3(1+b^2)$, we obtain

$$\begin{aligned} D_0 &= \int_0^\infty \frac{u}{\rho^2}a^{-\frac{5}{2}}e^{-\frac{c(a-1)}{2}}du \\ &= \int_0^\infty \left(1-\frac{cM}{\rho}\right)\sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}}e^{-\frac{cu}{\rho}}e^{-\frac{cN}{\rho u}}du \\ &= \left(1-\frac{cM}{\rho}\right)\underbrace{\int_0^\delta \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}}e^{-\frac{cu}{\rho}}e^{-\frac{cN}{\rho u}}du}_{D_1} \\ &+ \left(1-\frac{cM}{\rho}\right)\underbrace{\int_\delta^\infty \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}}e^{-\frac{cu}{\rho}}e^{-\frac{cN}{\rho u}}du}_{D_2} \end{aligned} \tag{F.69}$$

Applying Lemma 1 on D_1,

$$\begin{aligned}
D_1 &= \int_0^\delta \left(1-\frac{cu}{\rho}\right)\sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}}e^{-\frac{cN}{\rho u}}du \\
&= \underbrace{\int_0^\delta \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}}e^{-\frac{cN}{\rho u}}du}_{D_{11}} \\
&\quad - \underbrace{\int_0^\delta \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}}\frac{cu}{\rho}e^{-\frac{cN}{\rho u}}du}_{D_{12}}
\end{aligned}$$

Replace u with $t=\frac{cN}{\rho u}$ in D_{11} and D_{12}, so that $u=\frac{cN}{\rho t}$ and $du=\frac{cN}{\rho}(-1/t^2)dt$. So D_{11} changes into

$$\begin{aligned}
D_{11} &= \int_0^\delta \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}}e^{-\frac{cN}{\rho u}}du \\
&= \int_{\frac{cN}{\rho\delta}}^\infty \sqrt{\frac{(\frac{cN}{\rho t})^7\rho}{(2(\frac{cN}{\rho t})^2+(2M+\rho)(\frac{cN}{\rho t})+2N)^5}}e^{-t}\frac{cN}{\rho}(1/t^2)dt \\
&= \int_{\frac{cN}{\rho\delta}}^\infty \sqrt{\frac{(cN)^9\rho^2}{t(2(cN)^2+(2M+\rho)cN\rho t+2N\rho^2t^2)^5}}e^{-t}dt \\
&= \int_{\frac{cN}{\rho\delta}}^\infty O(\rho^{-4})e^{-t}dt \\
&= O(\rho^{-4})
\end{aligned}$$

With the same replacement, D_{12} becomes

$$\begin{aligned} D_{12} &= \int_0^{\delta} \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}} e^{-\frac{cN}{\rho u}} \frac{cu}{\rho} du \\ &= \int_{\frac{cN}{\rho\delta}}^{\infty} \sqrt{\frac{(\frac{cN}{\rho t})^7\rho}{(2(\frac{cN}{\rho t})^2+(2M+\rho)(\frac{cN}{\rho t})+2N)^5}} e^{-t} \frac{c^2N}{\rho^2 t} \frac{cN}{\rho} (1/t^2) dt \\ &= \int_{\frac{cN}{\rho\delta}}^{\infty} \sqrt{c^2 \frac{c(cN)^{10}}{\rho^2 t^2 t (2(cN)^2+(2M+\rho)cN\rho t+2N\rho^2 t^2)^5}} e^{-t} dt \\ &= \int_{\frac{cN}{\rho\delta}}^{\infty} O(\rho^{-6}) e^{-t} dt \\ &= O(\rho^{-6}) \end{aligned}$$

Therefore,

$$D_1 = O(\rho^{-4}) \tag{F.70}$$

Also applying lemma 1 to D_2,

$$\begin{aligned} D_2 &= \int_{\delta}^{\infty} \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}} e^{-\frac{cu}{\rho}} e^{-\frac{cN}{\rho u}} du \\ &= \int_{\delta}^{\infty} \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}} e^{-\frac{cu}{\rho}} (1-\frac{cN}{\rho u}) du \\ &= \underbrace{\int_{\delta}^{\infty} \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}} e^{-\frac{c}{\rho}} du}_{D_{21}} \\ &\quad - \underbrace{\int_{\delta}^{\infty} \frac{cN}{\rho u} \sqrt{\frac{u^7\rho}{(2u^2+(2M+\rho)u+2N)^5}} e^{-\frac{cu}{\rho}} du}_{D_{22}} \end{aligned} \tag{F.71}$$

Replace u with $t = \frac{cu}{\rho}$ in D_{21} and D_{22}, so that $u = \frac{\rho t}{c}$ and $du = \frac{\rho}{c}dt$, D_{21} changes into

$$\begin{aligned}
D_{21} &= \int_{\frac{c\delta}{\rho}}^{\infty} \sqrt{\frac{(\frac{\rho t}{c})^7 \rho}{(2(\frac{\rho t}{c})^2 + (2M+\rho)(\frac{\rho t}{c}) + 2N)^5}} e^{-t} \frac{\rho}{c} dt \\
&= \int_{\frac{c\delta}{\rho}}^{\infty} \sqrt{\frac{(\frac{\rho^{10} t^7}{c^9})}{(2(\frac{\rho t}{c})^2 + (2M+\rho)(\frac{\rho t}{c}) + 2N)^5}} e^{-t} dt \\
&= \int_{\frac{c\delta}{\rho}}^{\infty} \sqrt{\frac{(c\rho^{10} t^7)}{(2(\rho t)^2 + (2M+\rho)(c\rho t) + 2c^2 N)^5}} e^{-t} dt \\
&= \int_{\frac{c}{\rho}}^{\infty} \sqrt{\frac{ct^7}{(2t^2 + ct + 2Mct/\rho + 2c^2 N/\rho^2)^5}} e^{-t} dt
\end{aligned}$$

Lemma 8

$$\begin{aligned}
\frac{1}{\sqrt{f(t) + O(\rho^{-2})}} &= \frac{1}{\sqrt{f(t)}} - \left(\frac{1}{\sqrt{f(t)}} - \frac{1}{\sqrt{f(t) + O(\rho^{-2})}} \right) \\
&= \frac{1}{\sqrt{f(t)}} - \frac{\sqrt{f(t) + O(\rho^{-2})} - \sqrt{f(t)}}{\sqrt{f(t)}\sqrt{f(t) + O(\rho^{-2})}} \\
&= \frac{1}{\sqrt{f(t)}} - \frac{O(\rho^{-2})}{\sqrt{f(t)}\sqrt{f(t) + O(\rho^{-2})}(\sqrt{f(t) + O(\rho^{-2})} + \sqrt{f(t)})} \\
&= \frac{1}{\sqrt{f(t)}} + O(\rho^{-2})
\end{aligned}$$

Applying Eq. (F.50) in Lemma 5 and Lemma 8 on D_{21}, it becomes

$$\begin{aligned}
D_{21} &= \int_{\frac{c}{\rho}}^{\infty} \sqrt{\frac{ct^2}{(2t + c + 2Mc/\rho)^5}} e^{-t} dt + O(\frac{1}{\rho^2}) \\
&= \int_{\frac{c}{\rho}}^{\infty} \sqrt{c} t (\frac{1}{\sqrt{2t + c + 2Mc/\rho}})^5 e^{-t} dt + O(\frac{1}{\rho^2})
\end{aligned}$$

Applying Eq. (F.50) in Lemma 5 again and binomial series,

$$\begin{aligned}
D_{21} &= \int_{\frac{c}{\rho}}^{\infty} \sqrt{c}t(\frac{1}{\sqrt{2t+c}\sqrt{1+\frac{2Mc}{\rho(2t+c)}}})^5 e^{-t}dt + O(\frac{1}{\rho^2}) \\
&= \int_{\frac{c}{\rho}}^{\infty} \sqrt{\frac{c}{(2t+c)^5}}t(1-\frac{5}{2}\frac{2Mc}{\rho(2t+c)})e^{-t}dt + O(\frac{1}{\rho^2}) \\
&= \int_{\frac{c}{\rho}}^{\infty} \sqrt{\frac{c}{(2t+c)^5}}t(1-\frac{5Mc}{\rho(2t+c)})e^{-t}dt + O(\frac{1}{\rho^2}) \\
&= \underbrace{\int_{\frac{c}{\rho}}^{\infty} \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt}_{D_3} - \underbrace{\int_{\frac{c}{\rho}}^{\infty} t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt}_{D_4} + O(\frac{1}{\rho^2})
\end{aligned}$$

$$\begin{aligned}
D_3 &= \int_{\frac{c}{\rho}}^{\infty} \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt \\
&= \int_{0}^{\infty} \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt - \int_{0}^{\frac{c}{\rho}} \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt
\end{aligned}$$

Applying Lemma 5 on D_3,

$$\begin{aligned}
D_3 &= \int_{0}^{\infty} \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt - \int_{0}^{\frac{c}{\rho}} \frac{1}{c^2}(1+\frac{2t}{c})^{-\frac{5}{2}}te^{-t}dt \\
&= \int_{0}^{\infty} \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt - \int_{0}^{\frac{c}{\rho}} \frac{1}{c^2}(1-\frac{5t}{c})^{-\frac{5}{2}}te^{-t}dt \\
&= \int_{0}^{\infty} \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt - \left[\frac{1}{c^2}\int_{0}^{\frac{c}{\rho}} te^{-t}dt - \frac{5}{c^3}\int_{0}^{\frac{c}{\rho}} t^2e^{-t}dt\right] + O(\frac{1}{\rho^2})
\end{aligned}$$

Applying Lemma 3 on D_3,

$$
\begin{aligned}
D_3 &= \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}} te^{-t}dt - \left[\frac{1}{c^2}\int_0^{\frac{c}{\rho}} te^{-t}dt - \frac{5}{c^3}\left(\int_0^{\frac{c}{\rho}} 2te^{-t}dt\right)\right] + O(\frac{1}{\rho^2}) \\
&= \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}} te^{-t}dt - \frac{c-10}{c^3}\left[\int_0^{\frac{c}{\rho}} te^{-t}dt\right] + O(\frac{1}{\rho^2}) \\
&= \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}} te^{-t}dt + \frac{c-10}{c^3}\left[te^{-t}\Big|_0^{\frac{c}{\rho}} - \int_0^{\frac{c}{\rho}} e^{-t}dt\right] + O(\frac{1}{\rho^2}) \\
&= \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}} te^{-t}dt + \frac{c-10}{c^3}\left[\frac{c}{\rho}e^{-\frac{c}{\rho}} + e^{-t}\Big|_0^{\frac{c}{\rho}}\right] + O(\frac{1}{\rho^2}) \\
&= \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}} te^{-t}dt + \frac{c-10}{c^3}\left[\frac{c}{\rho}e^{-\frac{c}{\rho}} + e^{-\frac{c}{\rho}} - 1\right] + O(\frac{1}{\rho^2})
\end{aligned}
$$

Applying Lemma 1 on D_3,

$$
\begin{aligned}
D_3 &= \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}} te^{-t}dt + \frac{c-10}{c^3}\left[\frac{c}{\rho}(1-\frac{c}{\rho}) + (1-\frac{c}{\rho}) - 1\right] + O(\frac{1}{\rho^2}) \\
&= \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}} te^{-t}dt + O(\frac{1}{\rho^2})
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
D_4 &= \int_{\frac{c}{\rho}}^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt \\
&= \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt - \int_0^{\frac{c}{\rho}} t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt \\
&= \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt - \frac{5Mc^{\frac{3}{2}}}{\rho}\int_0^{\frac{c}{\rho}} t\frac{1}{(2t+c)^{\frac{7}{2}}}e^{-t}dt
\end{aligned}
$$

Applying Lemma 5 binomial series on D_4,

$$
\begin{aligned}
D_4 &= \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt - \frac{5Mc^{\frac{3}{2}}}{\rho}\int_0^{\frac{c}{\rho}} \frac{t}{c^{\frac{7}{2}}}\left(1-\frac{7}{2}\frac{2t}{c}\right)e^{-t}dt \\
&= \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt - \frac{5Mc^{\frac{3}{2}}}{\rho}\int_0^{\frac{c}{\rho}} \frac{t}{c^{\frac{7}{2}}}\left(1-\frac{7t}{c}\right)e^{-t}dt
\end{aligned}
$$

Applying Lemma 3 on D_4,

$$
\begin{aligned}
D_4 &= \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt - \frac{5Mc^{\frac{3}{2}}}{\rho}O(\frac{1}{\rho}) \\
&= \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt + O(\frac{1}{\rho^2})
\end{aligned}
$$

Therefore,

$$
D_{21} = \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt + O(\frac{1}{\rho^2}) - \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt + O(\frac{1}{\rho^2}) \quad \text{(F.72)}
$$

In Eq. (F.71), applying replacement on D_{22}, we have

$$
\begin{aligned}
D_{22} &= \int_{\frac{c\delta}{\rho}}^\infty \frac{c^2N}{\rho^2 t}\sqrt{\frac{(\frac{\rho t}{c})^7\rho}{(2(\frac{\rho t}{c})^2+(2M+\rho)(\frac{\rho t}{c})+2N)^5}}e^{-t}\frac{\rho}{c}dt \\
&= \int_{\frac{c\delta}{\rho}}^\infty O(\rho^{-2})e^{-t}dt \\
&= O(\rho^{-2}) \quad \text{(F.73)}
\end{aligned}
$$

Substituting Eqs. (F.72) and (F.73) into Eq. (F.71), we obtain

$$
D_2 = \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt - \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt + O(\frac{1}{\rho^2}) \quad \text{(F.74)}
$$

Substituting Eqs. (F.70) and (F.74) into Eq. (F.69), we obtain

$$\begin{aligned}
D_0 &= \left(1-\frac{cM}{\rho}\right)\left[\int_0^\infty \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt - \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt + O(\frac{1}{\rho^2})\right] \\
&= \int_0^\infty \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt - \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt - \frac{cM}{\rho}\int_0^\infty \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt
\end{aligned}$$

Substituting D_0 into Eq. (F.68), we obtain

$$\begin{aligned}
D = \frac{9}{\pi}\int_0^\infty \frac{1}{c}e^{-\frac{c}{2}}\Bigg[&\int_0^\infty \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt - \int_0^\infty t\frac{5Mc^{\frac{3}{2}}}{\rho(2t+c)^{\frac{7}{2}}}e^{-t}dt \\
&-\frac{cM}{\rho}\int_0^\infty \sqrt{\frac{c}{(2t+c)^5}}te^{-t}dt\Bigg]db
\end{aligned}$$

where $c = 3(1+b^2)$.

Let $t = \frac{c}{2}u$, so $dt = \frac{c}{2}du$, D becomes

$$\begin{aligned}
D &= \frac{9}{\pi}\int_0^\infty \frac{1}{c}e^{-\frac{c}{2}}\Bigg[\int_0^\infty \sqrt{\frac{c}{(2\frac{c}{2}u+c)^5}}\frac{c}{2}ue^{-\frac{c}{2}u}\frac{c}{2}du - \int_0^\infty \frac{c}{2}u\frac{5Mc^{\frac{3}{2}}}{\rho(2\frac{c}{2}u+c)^{\frac{7}{2}}}e^{-\frac{c}{2}u}\frac{c}{2}du \\
&-\frac{cM}{\rho}\int_0^\infty \sqrt{\frac{c}{(2\frac{c}{2}u+c)^5}}\frac{c}{2}ue^{-\frac{c}{2}u}\frac{c}{2}du\Bigg]db \\
&= \frac{9}{\pi}\int_0^\infty \frac{1}{c}e^{-\frac{c}{2}}\Bigg[\int_0^\infty \frac{1}{c^2}(1+u)^{-\frac{5}{2}}\frac{c}{2}ue^{-\frac{c}{2}u}\frac{c}{2}du - \int_0^\infty \frac{c}{2}u\frac{5M}{\rho}\frac{1}{c^2}(1+u)^{-\frac{7}{2}}e^{-\frac{c}{2}u}\frac{c}{2}du \\
&-\frac{cM}{\rho}\int_0^\infty \frac{1}{c^2}(1+u)^{-\frac{5}{2}}\frac{c}{2}ue^{-\frac{c}{2}u}\frac{c}{2}du\Bigg]db \\
&= \frac{9}{\pi}\int_0^\infty \frac{1}{c}e^{-\frac{c}{2}}\Bigg[\int_0^\infty (1+u)^{-\frac{5}{2}}\frac{1}{4}ue^{-\frac{c}{2}u}du - \int_0^\infty \frac{1}{4}u\frac{5M}{\rho}(1+u)^{-\frac{7}{2}}e^{-\frac{c}{2}u}du \\
&-\frac{cM}{\rho}\int_0^\infty \frac{1}{4}(1+u)^{-\frac{5}{2}}ue^{-\frac{c}{2}u}du\Bigg]db
\end{aligned}$$

Replace c with $c = 3(1+b^2)$,

$$D = \frac{3}{4\pi}\left[\underbrace{\int_0^\infty \frac{u}{(1+u)^{\frac{5}{2}}}\int_0^\infty \frac{e^{-\frac{3}{2}(1+b^2)(1+u)}}{1+b^2}dbdu}_{E_1} - \underbrace{\int_0^\infty \frac{5M}{\rho}\frac{u}{(1+u)^{\frac{7}{2}}}\int_0^\infty \frac{e^{-\frac{3}{2}(1+b^2)(1+u)}}{1+b^2}dbdu}_{E_2}\right.$$
$$\left. - \underbrace{\frac{3M}{\rho}\int_0^\infty \frac{u}{(1+u)^{\frac{5}{2}}}\int_0^\infty e^{-\frac{3}{2}(1+b^2)(1+u)}dbdu}_{E_3}\right] \tag{F.75}$$

$$E_1 = \int_0^\infty \frac{u}{(1+u)^{\frac{5}{2}}}\int_0^\infty e^{-\frac{3}{2}(1+b^2)(1+u)}dbdu$$

Let $b = \cot\theta$, so $db = (-1-\cot^2\theta)d\theta$,

$$\begin{aligned}E_1 &= \int_0^\infty \frac{u}{(1+u)^{\frac{5}{2}}}\left[\int_0^{\frac{\pi}{2}} \frac{e^{-\frac{3}{2}\frac{1}{\sin^2\theta}(1+u)}}{1+\cot^2\theta}(1+\cot^2\theta)d\theta\right]du \\ &= \pi\int_0^\infty \frac{u}{(1+u)^{\frac{5}{2}}}Q(\sqrt{3(1+u)})du \\ &= \pi/2\int_0^\infty \frac{u}{(1+u)^{\frac{5}{2}}}\mathrm{erfc}(\sqrt{3(1+u)/2})du\end{aligned}$$

Applying integration by parts, E_1 becomes

$$\begin{aligned}E_1 &= \pi/2(-2/3)\int_0^\infty u\mathrm{erfc}(\sqrt{3(1+u)/2})d(1+u)^{-\frac{3}{2}} \\ &= -\pi/3\left[u\mathrm{erfc}(\sqrt{3(1+u)/2})(1+u)^{-\frac{3}{2}}\,|_0^\infty - \int_0^\infty (1+u)^{-\frac{3}{2}}d[u\mathrm{erfc}(\sqrt{3(1+u)/2})]\right]\end{aligned} \tag{F.76}$$

Lemma 9: Leibniz integral rule

Leibniz integral rule is about derivative of integral, when the limits of integration a and b and the integrand (x, α) all are functions of the parameter α, the formula is:

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx = \frac{db(\alpha)}{d\alpha} f(b(\alpha),\alpha) - \frac{da(\alpha)}{d\alpha} f(a(\alpha),\alpha) + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx \tag{F.77}$$

Applying Leibniz integral rule,

$$\begin{aligned}
&\frac{d(\mathrm{erfc}(\sqrt{3(1+u)/2})}{du} \\
=&\frac{d(\frac{2}{\sqrt{\pi}}\int_{\sqrt{\frac{3(1+u)}{2}}}^{\infty} e^{-t^2}dt)}{du} \\
=&-\frac{2}{\sqrt{\pi}}e^{-\frac{3(1+u)}{2}}\frac{d(\sqrt{\frac{3(1+u)}{2}})}{du} \\
=&-\frac{2}{\sqrt{\pi}}e^{-\frac{3(1+u)}{2}}\frac{1}{2}\left(\frac{3(1+u)}{2}\right)^{-\frac{1}{2}}\frac{3}{2} \\
=&-\sqrt{\frac{3}{2\pi}}e^{-\frac{3(1+u)}{2}}(1+u)^{-\frac{1}{2}}
\end{aligned} \tag{F.78}$$

Therefore, from Eq. (F.78),

$$\begin{aligned}
\frac{d[u\mathrm{erfc}(\sqrt{3(1+u)/2})]}{du} &= \mathrm{erfc}(\sqrt{3(1+u)/2}) + u\frac{d(\mathrm{erfc}(\sqrt{3(1+u)/2})}{du} \\
&= \mathrm{erfc}(\sqrt{3(1+u)/2}) - \sqrt{\frac{3}{2\pi}}e^{-\frac{3(1+u)}{2}}u(1+u)^{-\frac{1}{2}}
\end{aligned} \tag{F.79}$$

Substituting Eq. (3.6) into Eq. (F.76), it becomes

$$\begin{aligned} E_1 &= \pi/3 \int_0^{\infty} (1+u)^{-\frac{3}{2}} \left[\mathrm{erfc}(\sqrt{3(1+u)/2}) - \sqrt{\frac{3}{2\pi}} e^{-\frac{3(1+u)}{2}} u(1+u)^{-\frac{1}{2}} \right] du \\ &= \pi/3 \left[-2 \int_0^{\infty} \mathrm{erfc}(\sqrt{3(1+u)/2}) d(1+u)^{-\frac{1}{2}} \right] - \pi/2 \left[\sqrt{\frac{2}{3\pi}} \int_0^{\infty} \frac{u}{(1+u)^2} e^{-\frac{3(1+u)}{2}} du \right] \end{aligned}$$

Applying integration by parts, E_1 becomes

$$\begin{aligned} E_1 &= \pi/3 \left[-2 \left(\mathrm{erfc}(\sqrt{3(1+u)/2})(1+u)^{-\frac{1}{2}} \Big|_0^{\infty} - \int_0^{\infty} (1+u)^{-\frac{1}{2}} d\mathrm{erfc}(\sqrt{3(1+u)/2}) \right) \right] \\ &- \pi/2 \left[\sqrt{\frac{2}{3\pi}} \int_0^{\infty} \left(\frac{1}{(1+u)} - \frac{1}{(1+u)^2} \right) e^{-\frac{3(1+u)}{2}} du \right] \\ &= \pi/2 \left[\frac{4}{3} \mathrm{erfc}(\sqrt{3/2}) + \sqrt{\frac{2}{3\pi}} e^{-\frac{3}{2}} - \left(3\sqrt{\frac{3}{2\pi}} \right) \int_0^{\infty} \frac{e^{-\frac{3(1+u)}{2}}}{1+u} du \right] \end{aligned} \quad \text{(F.80)}$$

If define $t = \frac{3}{2}u$, we have $dt = \frac{3}{2}du$,

$$\int_0^{\infty} \frac{e^{-\frac{3(1+u)}{2}}}{1+u} du = e^{-\frac{3}{2}} \int_0^{\infty} \frac{e^{-t}}{t+\frac{3}{2}} dt \quad \text{(F.81)}$$

Substituting Eq. (F.81) into Eq. (F.80), we have

$$E_1 = \pi/2 \left[\frac{4}{3} \mathrm{erfc}(\sqrt{3/2}) + \sqrt{\frac{2}{3\pi}} e^{-\frac{3}{2}} - \sqrt{\frac{27}{2\pi}} \int_0^{\infty} e^{-\frac{3}{2}} \int_0^{\infty} \frac{e^{-t}}{t+\frac{3}{2}} dt \right] \quad \text{(F.82)}$$

Similarly,

$$
\begin{aligned}
E_2 &= \frac{5M}{\rho}\frac{\pi}{2}\left(-\frac{2}{5}\right)\int_0^\infty u\mathrm{erfc}(\sqrt{3(1+u)/2})d(1+u)^{-\frac{5}{2}} \\
&= \frac{M\pi}{\rho}\left[-\frac{2}{3}\int_0^\infty \mathrm{erfc}(\sqrt{3(1+u)/2})d(1+u)^{-\frac{3}{2}}\right] - \frac{M\pi}{\rho}\sqrt{\frac{3}{2\pi}}\int_0^\infty \frac{u}{(1+u)^3}e^{-\frac{3(1+u)}{2}}du
\end{aligned} \tag{F.83}
$$

In E_3, let $t = \frac{3}{2}(1+u)b^2$,

$$
\begin{aligned}
E_3 &= \frac{3M}{\rho}\sqrt{\frac{2}{3}}\int_0^\infty \frac{u}{(1+u)^3}e^{-\frac{3(1+u)}{2}}\left[\int_0^\infty e^{-t^2}dt\right]du \\
&= \frac{M\pi}{\rho}\sqrt{\frac{3}{2\pi}}\int_0^\infty \frac{u}{(1+u)^3}e^{-\frac{3(1+u)}{2}}du
\end{aligned} \tag{F.84}
$$

Therefore,

$$
\begin{aligned}
E_2 + E_3 &= \frac{M\pi}{\rho}\left[-\frac{2}{3}\int_0^\infty \mathrm{erfc}(\sqrt{3(1+u)/2})d(1+u)^{-\frac{3}{2}}\right] \\
&= \frac{M\pi}{\rho}\left[2\mathrm{erfc}(\sqrt{\frac{3}{2}}) + \sqrt{\frac{2}{3\pi}}e^{-\frac{3}{2}}\int_0^\infty \frac{e^{-t}}{t+\frac{3}{2}}dt\right]
\end{aligned} \tag{F.85}
$$

Substituting Eqs. (F.82) and (F.85) back into Eq. (F.75), we have

$$\begin{aligned} D &= \frac{3}{8}\left[\frac{4}{3}\text{erfc}(\sqrt{3/2}) + \sqrt{\frac{2}{3\pi}}e^{-\frac{3}{2}} - \sqrt{\frac{27}{2\pi}}\int_0^\infty e^{-\frac{3}{2}}\int_0^\infty \frac{e^{-t}}{t+\frac{3}{2}}dt\right] \\ &- \frac{3}{4\pi}\frac{M\pi}{\rho}\left[2\text{erfc}(\sqrt{\frac{3}{2}}) + \sqrt{\frac{2}{3\pi}}e^{-\frac{3}{2}}\int_0^\infty \frac{e^{-t}}{t+\frac{3}{2}}dt\right] \\ &= \frac{1}{2}\text{erfc}(\sqrt{3/2}) + \frac{1}{4}\sqrt{\frac{3}{2\pi}}e^{-\frac{3}{2}} - \frac{3}{8}\sqrt{\frac{27}{2\pi}}\int_0^\infty e^{-\frac{3}{2}}\int_0^\infty \frac{e^{-t}}{t+\frac{3}{2}}dt \\ &- \frac{3M}{4\rho}\left[2\text{erfc}(\sqrt{\frac{3}{2}}) + \sqrt{\frac{2}{3\pi}}e^{-\frac{3}{2}}\int_0^\infty \frac{e^{-t}}{t+\frac{3}{2}}dt\right] \end{aligned} \tag{F.86}$$

Above all, substituting Eqs. (F.39), (F.59), (F.64) and (F.86) into Eq. (F.27),

$$P_e(\mathbf{F}) = A - B - C - D \tag{F.87}$$

$$\begin{aligned} &= \frac{1}{2}\text{erfc}(\frac{\sqrt{6}}{2}) - \frac{3M}{2}\text{erfc}(\frac{\sqrt{6}}{2})\rho^{-1} + O(\rho^{-2}) \\ &- \sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}\left[-\frac{1}{4} + \frac{5}{8}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt + \left(-\frac{M}{4} - \frac{3M}{8}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt\right)\rho^{-1}\right] \\ &- \sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}\left[\frac{1}{6} - \frac{1}{4}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt - \frac{5M}{4}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt\rho^{-1}\right] \\ &- \frac{1}{2}\text{erfc}(\sqrt{3/2}) - \frac{1}{4}\sqrt{\frac{3}{2\pi}}e^{-\frac{3}{2}} + \frac{3}{8}\sqrt{\frac{27}{2\pi}}\int_0^\infty e^{-\frac{3}{2}}\int_0^\infty \frac{e^{-t}}{t+\frac{3}{2}}dt \\ &+ \frac{3M}{4}\left[2\text{erfc}(\sqrt{\frac{3}{2}}) + \sqrt{\frac{2}{3\pi}}e^{-\frac{3}{2}}\int_0^\infty \frac{e^{-t}}{t+\frac{3}{2}}dt\right]\rho^{-1} \end{aligned} \tag{F.88}$$

Therefore in Eq. (F.87), the constant part is

$$\begin{aligned}
\text{Const} &= \frac{1}{2}\text{erfc}(\frac{\sqrt{6}}{2}) - \sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}\left[-\frac{1}{12} + \frac{3}{8}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt\right] \\
&- \frac{1}{2}\text{erfc}(\sqrt{3/2}) - \frac{1}{4}\sqrt{\frac{3}{2\pi}}e^{-\frac{3}{2}} + \frac{3}{8}\sqrt{\frac{27}{2\pi}}\int_0^\infty e^{-\frac{3}{2}}\int_0^\infty \frac{e^{-t}}{t+\frac{3}{2}}dt \\
&= 0
\end{aligned}$$

In Eq. (F.87), the $O(\rho^{-1})$ term,

$$\begin{aligned}
O(\rho^{-1}) &= -\frac{3M}{2}\text{erfc}(\frac{\sqrt{6}}{2})\rho^{-1} - \sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}\left[-\frac{M}{4} - \frac{3M}{8}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt\right. \\
&\left. -\frac{5M}{4}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt\right]\rho^{-1} + \frac{3M}{4}\left[2\text{erfc}(\sqrt{\frac{3}{2}}) + \sqrt{\frac{2}{3\pi}}e^{-\frac{3}{2}}\int_0^\infty \frac{e^{-t}}{t+\frac{3}{2}}dt\right]\rho^{-1} \\
&= \left[\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}\frac{M}{4} + \left(\frac{13M}{8}\sqrt{\frac{27}{2\pi}} + \frac{3M}{4}\sqrt{\frac{2}{3\pi}}\right)e^{-\frac{3}{2}}\int_0^\infty \frac{1}{t+3/2}e^{-t}dt\right]\rho^{-1} \\
&= \left[\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}\frac{M}{4} + \frac{43M}{8}\sqrt{\frac{3}{2\pi}}\int_0^\infty \frac{1}{t+3/2}e^{-(t+\frac{3}{2})}dt\right]\rho^{-1} \\
&= \left[\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}\frac{M}{4} + \frac{43M}{8}\sqrt{\frac{3}{2\pi}}\int_{\frac{3}{2}}^\infty \frac{1}{m}e^{-m}dm\right]\rho^{-1} \\
&= \left[\sqrt{\frac{27}{2\pi}}e^{-\frac{3}{2}}\frac{M}{4} + \frac{43M}{8}\sqrt{\frac{3}{2\pi}}E_1\left(\frac{3}{2}\right)\right]\rho^{-1} \\
&= C_{-1}M\rho^{-1}
\end{aligned}$$

where $m = t + \frac{3}{2}$ and $C_{-1} = \frac{3}{4}\sqrt{\frac{3}{2\pi}}e^{-\frac{3}{2}} + \frac{43}{8}\sqrt{\frac{3}{2\pi}}E_1\left(\frac{3}{2}\right)$ which is positive, i.e. $C_{-1} > 0$.

Therefore, the asymptotic formula has the following form

$$P_e(\mathbf{F}) = C_{-1}M\rho^{-1} + O(\rho^{-2}) \qquad \text{(F.89)}$$

where $M = d_1^{-1} + d_2^{-1}$.

Bibliography

[1] Y. Ding, T. Davidson, Z.-Q. Luo, and K. Wong, "Minimum ber block precoders for zero-forcing equalization," *IEEE Transactions on Signal Processing*, vol. 51, no. 9, pp. 2410–2423, 2003.

[2] A. Paulraj and T. Kailath, "Increasing capacity in wireless broadcast systems using distributed transmission/directional reception (dtdr)," Sep. 6 1994, uS Patent 5,345,599.

[3] G. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell labs technical journal*, vol. 1, no. 2, pp. 41–59, 1996.

[4] G. Golden, C. Foschini, R. Valenzuela, and P. Wolniansky, "Detection algorithm and initial laboratory results using v-blast space-time communication architecture," *Electronics letters*, vol. 35, no. 1, pp. 14–16, 1999.

[5] V. Tarokh, N. Seshadri, and A. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Transactions on Information Theory*, vol. 44, no. 2, pp. 744–765, 1998.

[6] S. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE Journal on Selected Areas in Communications*, vol. 16, no. 8, pp. 1451–1458, 1998.

[7] V. Tarokh, H. Jafarkhani, and A. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Transactions on Information Theory*, vol. 45, no. 5, pp. 1456–1467, 1999.

[8] G. Foschini and M. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless personal communications*, vol. 6, no. 3, pp. 311–335, 1998.

[9] E. Telatar, "Capacity of multi-antenna gaussian channels," *European Transactions on Telecommunications*, vol. 10, no. 6, pp. 585–595, 1999.

[10] L. J. Zhang, J.-K. and K. Wong, *Space-time processing for MIMO communications*. Wiley Online Library, 2005.

[11] H. El Gamal and M. Damen, "Universal space-time coding," *IEEE Transactions on Information Theory*, vol. 49, no. 5, pp. 1097–1119, 2003.

[12] X. Ma and G. Giannakis, "Full-diversity full-rate complex-field space-time coding," *IEEE Transactions on Signal Processing*, vol. 51, no. 11, pp. 2917–2930, 2003.

[13] A. Scaglione, G. Giannakis, and S. Barbarossa, "Redundant filterbank precoders and equalizers. ii. blind channel estimation, synchronization, and direct equalization," *IEEE Transactions on Signal Processing*, vol. 47, no. 7, pp. 2007–2022, 1999.

[14] C. Wu and S. Chern, "Minimum ber block-based precoder design for zero-forcing equalization: An oblique projection framework," *IEEE Transactions on Signal Processing*, vol. 55, no. 12, pp. 5630–5642, 2007.

[15] J. Liu, J.-K. Zhang, and K. Wong, "On the design of minimum ber linear space-time block codes for mimo systems equipped with mmse receivers," *IEEE Transactions on Signal Processing*, vol. 54, no. 8, pp. 3147–3158, 2006.

[16] D. Palomar, J. Cioffi, and M. Lagunas, "Joint tx-rx beamforming design for multicarrier mimo channels: A unified framework for convex optimization," *IEEE Transactions on Signal Processing*, vol. 51, no. 9, pp. 2381–2401, 2003.

[17] A. Scaglione, S. Barbarossa, and G. Giannakis, "Filterbank transceivers optimizing information rate in block transmissions over dispersive channels," *IEEE Transactions on Information Theory*, vol. 45, no. 3, pp. 1019–1032, 1999.

[18] S. Chan, T. Davidson, and K. Wong, "Asymptotically minimum ber linear block precoders for mmse equalisation," in *IEE Proceedings-Communications*, vol. 151, no. 4. IET, 2004, pp. 297–304.

[19] J. Kermoal, L. Schumacher, K. Pedersen, P. Mogensen, and F. Frederiksen, "A stochastic mimo radio channel model with experimental validation," *IEEE Journal on Selected Areas in Communications*, vol. 20, no. 6, pp. 1211–1226, 2002.

[20] K. Yu, M. Bengtsson, B. Ottersten, D. McNamara, P. Karlsson, and M. Beach, "Modeling of wide-band mimo radio channels based on nlos indoor measurements," *IEEE Transactions on Vehicular Technology*, vol. 53, no. 3, pp. 655–665, 2004.

[21] A. Tulino, A. Lozano, and S. Verdú, "Impact of antenna correlation on the capacity of multiantenna channels," *IEEE Transactions on Information Theory*, vol. 51, no. 7, pp. 2491–2509, 2005.

[22] T.-T. Liu, J.-K. Zhang, and K. Wong, "Optimal precoder design for correlated mimo communication systems using zero-forcing decision feedback equalization," vol. 57, no. 9, pp. 3600–3612, 2009.

[23] R. Muirhead and I. ebrary, *Aspects of multivariate statistical theory.* Wiley Online Library, 1982.

[24] J.G.Proakis, *Digital Communications*, 4th ed. New York:McGrawHill, 2001.

[25] S. Boyd and L. Vandenberghe, *Convex optimization.* Cambridge Univ Pr, 2004.

[26] R. Horn and C. Johnson, *Topics in matrix analysis.* Cambridge Univ Pr, 1994.

[27] W. Smith and R. Hocking, "Algorithm AS 53: Wishart variate generator," *Applied Statistics*, vol. 21, no. 3, pp. 341–345, 1972.

[28] L. Zheng and D. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *Information Theory, IEEE Transactions on*, vol. 49, no. 5, pp. 1073–1096, 2003.

[29] J. Winters, J. Salz, and R. Gitlin, "The impact of antenna diversity on the capacity of wireless communication systems," *Communications, IEEE Transactions on*, vol. 42, no. 234, pp. 1740–1751, 1994.

[30] S. Loyka, "Channel capacity of MIMO architecture using the exponentialcorrelation matrix," *IEEE Communications letters*, vol. 5, no. 9, pp. 369–371, 2001.

[31] K. Cho and D. Yoon, “On the general ber expression of one-and two-dimensional amplitude modulations,” *Communications, IEEE Transactions on*, vol. 50, no. 7, pp. 1074–1080, 2002.