Inverse Sampling Procedures to Test for Homogeneity in a Multivariate Hypergeometric Distribution

INVERSE SAMPLING PROCEDURES TO TEST FOR HOMOGENEITY IN A MULTIVARIATE HYPERGEOMETRIC DISTRIBUTION

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A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND STATISTICS AND THE SCHOOL OF GRADUATE STUDIES OF MCMASTER UNIVERSITY

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

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Master of Science (2011) McMaster University
(Mathematics and Statistics) Hamilton, Ontario, Canada

TITLE: Inverse Sampling Procedures to Test for Homogeneity in a Multivariate Hypergeometric Distribution

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NUMBER OF PAGES: vi, 85

Abstract

In this thesis we study several inverse sampling procedures to test for homogeneity in a multivariate hypergeometric distribution. The procedures are finite population analogues of the procedures introduced in Panchapakesan et al. (1998) for the multinomial distribution. In order to develop some exact calculations for critical values not considered in Panchapakesan et al. we introduce some terminologies for target probabilities, transfer probabilities, potential target points, right intersection, and left union. Under the null and the alternative hypotheses, we give theorems to calculate the target and transfer probabilities, we then use these results to develop exact calculations for the critical values and powers of one of the procedures. We also propose a new approximate calculation. In order to speed up some of the calculations, we propose several fast algorithms for multiple summation.

The computing results showed that the simulations agree closely with the exact results. For small population sizes the critical values and powers of the procedures are different from the corresponding multinomial procedures, but when $N \geq 1680000$, all the results are the same as those in the multinomial distribution.

Acknowledgements

I thank my supervisor, Dr. Aaron Childs, for introducing me into the homogeneity testing and giving me the opportunity to work in a very interesting area, and for his support and guidance throughout my graduate studies at McMaster University.

His ideas, clues, recommendations and suggestions have been invaluable for the problems to be solved.

I also wish to thank professor Balakrishnan who has accepted, guided and taught me in statistics study.

I thank my wife, Mrs Daihong Chen, for her understanding and support to be a student again.

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1 Introduction

Hypothesis testing is to determine whether or not sample data strongly support an alternate hypothesis H_a about the population parameters. Normally if θ denotes a population parameter, the general format of the null and alternative hypothesis is $H_0 : \theta \in \Theta_0$ and $H_a : \theta \in \Theta_0^c$, where Θ_0 is a set of possible values of θ (see Casella and Berger (2002)). Homogeneity testing is used to test whether some parameters in a multivariate distribution are equal or not. For example in a multinomial distribution with parameters $p_1, p_2, ..., p_k$, we can test whether $p_1 = p_2 = ... = p_k$ or not. If we have a group of balls with different colours from a multivariate hypergeometric population, we can test whether the number of balls of each colour $M_1, M_2, ..., M_k$ are equal or not. Inverse sampling procedures are different from the traditional fixed sample size procedures. Fixed sample size procedures are based on a sample whose size is pre-determined. With inverse sampling procedures, observations are taken one at a time and the sampling is terminated as soon as a pre-specified condition is met. Often with inverse sampling procedures a very limited amount of data is necessary in order to make a decision whether to reject a hypothesis or not. Based on research of inverse sampling procedures to test for homogeneity in a multinomial distribution by Panchapakesan et al. (1998), here we study inverse sampling procedures to test for homogeneity in a multivariate hypergeometric distribution.

Let $(X_1, X_2, ..., X_k)$ be a random vector from a multinomial distribution with parameters $p_1, p_2, ..., p_k$ and let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(k)}$ be the order statistics in increasing order of magnitude. The null hypothesis is $p_1 = p_2 = ... = p_k$. In the inverse sampling Procedure 1 of Panchapakesan et al. (1998), observations are taken one at a time and the sampling is terminated when the count in any one category reaches a specified number M. The test statistic is $W_k = X_{(k)} - X_{(1)}$. We reject H_0 if $W_k \geq D_1$ where the critical values D_1 of W_k are determined so that the test has significance level α: $P_{H_0}(W_k \geq D_1) \leq \alpha$. In Procedure 2, observations are taken one at a time and after each observation we calculate $W_{kt} = X_{(k)t} - X_{(k-1)t}$, where t is the current sampling time. We reject H_0 if $W_{kt} \geq D_2$ for some $t \leq N_0$ (to be specified) where the critical values of D_2 are determined so that the test has significance level α:

$P_{H_0}(W_{kt} \geq D_2) \leq \alpha$. Powers of Procedures 1 or 2 are the probability of the test statistics being in the critical region $W_k \geq D_1$ or $W_{kt} \geq D_2$ under H_a.

Johnson and Young (1960) studied fixed sample size procedures based on

$$R_k = \frac{X_{(1)}}{X_{(k)}}$$

and

$$W_k = \frac{X_{(k)} - X_{(1)}}{n},$$

where n is a fixed sample size.Young (1962) considered similar procedures and compared the tests with the χ^2-test using large-sample approximations. Gupta and Nagel (1967) examined the mean and variance of the two extreme order statistics from a multinomial distribution.

The multinomial distribution is a special case of the multivariate hypergeometric distribution. When the total population size of a multivariate hypergeometric distribution is large enough, the multivariate hypergeometric distribution will become the multinomial distribution. So in this paper we generalize the procedures used in the research of inverse sampling procedures to test for homogeneity in a multinomial distribution by considering the population to be finite and the sampling to be without replacement. The above 2 procedures of Panchapakesan et al. (1998) and their combination will be used in this research.

In Chapter 2 we summarize the inverse sampling procedures to test for homogeneity in a multinomial distribution by Panchapakesan et al. (1998). In Chapter 3 we study calculation methods for obtaining the exact critical values of Procedure 1 and the approximate and exact critical values of Procedure 2 for the multivariate hypergeometric distribution, as well as a comparison of both methods with simulation. In Chapter 4 we develop calculation methods for the exact powers of Procedure 1 and the approximate and exact powers of Procedure 2 for the multivariate hypergeometric distribution and do a comparison with simulation. In Chapter 5 we calculate the exact and approximate power of the χ^2-test, and develop a fast algorithm

for these calculations. In Chapter 6 we research related results on the combination of the two procedures, and in Chapter 7 we give directions for future research.

2 Homogeneity Testing for the Multinomial Distribution

In this chapter we summarize the inverse sampling procedures to test for homogeneity in a multinomial distribution introduced by Panchapakesan et al. (1998). Before we began our new research we developed programs for the calculations in this previous research using the R and C languages to implement the calculations.

2.1 Hypothesis of Homogeneity

Suppose the random variables $X_1, X_2, ..., X_k$ are from a multinomial distribution with probability parameters $p_1, p_2, ..., p_k$. Then the joint probability mass function of $X_1, X_2, ..., X_k$ is given by

$$f(x_1, x_2, ..., x_k) = \binom{n}{x_1, x_2, ..., x_k} p_1^{x_1} p_2^{x_2} ... p_k^{x_k}$$
$$\sum_{i=1}^{k} x_i = n, \sum_{i=1}^{k} p_i = 1$$

The multinomial distribution provides a model for a population which is categorized into several classes. Such populations exist widely in ecology, sociology, genetics, economics, engineering and other disciplines. Are these categories homogeneous, i.e., with the same probability in each category? This is the null hypothesis of the homogeneity test for the multinomial distribution,

$$H_0 : p_1 = p_2 = ... = p_k.$$

The alternative hypothesis considered in Panchapakesan et al. (1998) is

$$H_a : p_1 = p_2 = ... = p_{k-1} = p \quad \text{and} \quad p_k = \lambda p, \lambda > 0,$$

with $\lambda = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 2, 3, 4, 5, 1$.

2.2 The Inverse Sampling Procedures

Let $X_1, X_2, ..., X_k$ be a random sample from the multinomial distribution and let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(k)}$ be the order statistics in increasing order of magnitude.

Procedure 1

Observations are taken one at a time and the sampling is terminated when the count in any category reaches a specified number M. Let $X_1, X_2, ..., X_k$ be the category counts at termination. Now make a simple calculation $W_k = X_{(k)} - X_{(1)}$. We reject H_0 if $W_k \geq D_1$, where the critical value D_1 is the smallest value that makes the significance level of the test less than or equal to α: $P_{H_0}(W_k \geq D_1) \leq \alpha$.

Procedure 2

Take one observation at a time and each time calculate $W_{kt} = X_{(k)t} - X_{(k-1)t}$, where t is the sampling time. If $W_{kt} \geq D_2$ for some $t \leq N_0$, where N_0 is a specified positive integer, stop sampling and reject H_0. If $W_{kt} < D_2$ for all $t \leq N_0$, then terminate sampling and accept H_0. The critical value D_2 is the smallest value that makes the significance level of the test less than or equal to α: $P_{H_0}(W_{kt} \geq D_2) \leq \alpha$.

The stopping rule of Procedure 1 and 2 have been used by Cacoullos and Sobel (1966) and Alam (1971), respectively, for selecting the most probable multinomial cell. A combination of the stopping rules was used by Ramey and Alam (1979) in their inverse sampling procedure for selecting the most probable cell.

2.3 The χ^2-Test

Under the null hypothesis, for a fixed sample size n, the random number of observations in

each category X_i has expected value $\frac{n}{k}$. Therefore a useful measure for the overall discrepancy in a sample is given by

$$T = \frac{k}{n}\sum_{i=1}^{k}(x_i - \frac{n}{k})^2$$

Under the null hypothesis T has an approximate chi-square distribution. Because a large value of the overall discrepancy indicates the rejection of the null hypothesis, the upper tail of the χ^2 distribution $\chi^2_{k-1,\alpha}$ constitutes the rejection region.

The powers of the above procedures can be compared with those of the χ^2-test as checking the effectiveness of the procedures. Details of the χ^2-test can be found in Rao (1973) and Hogg and Craig (1995).

2.4 Critical Values, Powers, and Sample Size

The critical values D_1 for Procedure 1 are obtained by finding the smallest integer D_1 such that $P_{H_0}(W_k \geq D_1) \leq \alpha$ for specified α, where

$$P_{H_0}(W_k \geq D_1) = 1 - \sum_{x_{k-1}=M-D_1+1}^{M-1} \cdots \sum_{x_1=M-D_1+1}^{M-1} \frac{(M-1+x_0)!p_1^{x_1}p_2^{x_2}...p_{k-1}^{x_{k-1}}p_k^M}{(M-1)!x_1!x_2!...x_{k-1}!},$$

and $x_0 = x_1 + x_2 + ... + x_{k-1}$.

The power of Procedure 1 is given by

$$\begin{aligned} P_{H_a}(W_k \geq D_1) = 1 - (k-1) & \sum_{x_2=M-D_1+1}^{M-1} \cdots \sum_{x_k=M-D_1+1}^{M-1} \frac{(M-1+x_2+...+x_k)!p^{M+x_2+...+x_{k-1}}p_k^{x_k}}{(M-1)!x_2!...x_k!} \\ & - \sum_{x_1=M-D_1+1}^{M-1} \cdots \sum_{x_{k-1}=M-D_1+1}^{M-1} \frac{(M-1+x_1+...+x_{k-1})!p^{x_1+...+x_{k-1}}p_k^M}{(M-1)!x_1!...x_{k-1}!} \end{aligned}$$

Power of the χ^2-test are obtained by

$$\beta_{\chi^2_{k-1,\alpha}}(n) = \sum_{\substack{x_1+\ldots+x_k=n \\ T\geq\chi^2_{k-1,\alpha}}} \cdots \sum \binom{n}{x_1,x_2,\ldots,x_k} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k},$$

where $T = \frac{k}{n}\sum_{i=1}^{k}(x_i - \frac{n}{k})^2$.

Since one minus the power of the procedures is the probability of Type 2 error, the power provides a method of checking the effectiveness of Procedures 1 and 2 for testing the null hypothesis. Comparison with the powers of the χ^2-test give us another way of checking the effectiveness of the procedures. Although we developed fast algorithms for the exact power calculation for the χ^2-test, it still takes too long when k grows bigger. So for $k > 7$, we used the approximation methods of Patnaik (1949) to calculate the powers of the χ^2-test. In the paper of Panchapakesan et al. (1998) the approximate methods were used for $k > 5$, so here we give the revised table for $k = 6$ and 7 using exact instead of the approximations in Table 12 of Panchapakesan et al. (1998).

Table 2.1 Exact powers of the χ^2-test at the 3% significance level for the multinomial distribution

k	n	λ								
		$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	2	3	4	5	1
6	29	0.1265	0.1095	0.0869	0.0567	0.1586	0.4860	0.7569	0.8993	0.0271
	35	0.1726	0.1457	0.1110	0.0673	0.1950	0.5776	0.8396	0.9480	0.0283
	41	0.2253	0.1867	0.1378	0.0787	0.2349	0.6648	0.9018	0.9762	0.0294
	47	0.2704	0.2197	0.1569	0.0839	0.2617	0.7230	0.9355	0.9880	0.0274
	53	0.3396	0.2727	0.1906	0.0971	0.3008	0.7830	0.9611	0.9945	0.0284
	59	0.4149	0.3308	0.2272	0.1106	0.3389	0.8320	0.9772	0.9976	0.0290
	65	0.4899	0.3912	0.2665	0.1251	0.3772	0.8711	0.9867	0.9990	0.0296
7	33	0.1110	0.0971	0.0784	0.0530	0.1545	0.4876	0.7665	0.9091	0.0273
	40	0.1405	0.1202	0.0937	0.0592	0.1852	0.5754	0.8469	0.9546	0.0269
	47	0.1823	0.1532	0.1157	0.0689	0.2218	0.6581	0.9040	0.9785	0.0278
	54	0.2256	0.1861	0.1364	0.0766	0.2571	0.7293	0.9424	0.9906	0.0276
	61	0.2817	0.2291	0.1637	0.0872	0.2940	0.7880	0.9658	0.9959	0.0282
	68	0.3398	0.2734	0.1913	0.0972	0.3299	0.8355	0.9800	0.9983	0.0283
	75	0.4051	0.3238	0.2228	0.1085	0.3659	0.8740	0.9887	0.9993	0.0284

Comparing the exact powers of the χ^2-test with the approximate results in Table 12 of Panchapakesan et al. (1998), we can see that the approximations are consistent with the exact results. And the results show that the smaller n is, the bigger the errors in the approximations are. For example, the error is 48.5% when $k = 7, \lambda = \frac{1}{5}$ and $n = 33$. The error is just 2.6% when $k = 7, \lambda = \frac{1}{5}$ and $n = 75$. Generally the approximations are comparable and reasonable as replacements for the exact results if the exact results require heavy computational effort and time.

Since the sample size or waiting time ω for either inverse sampling procedure is not fixed, the expected sample sizes were also calculated. For procedure 1, the expected sample size $E(\omega)$ can be obtained by the following formula:

$$\begin{aligned} Pr(\omega = M + j) \quad = \quad & \sum_{\substack{x_2+\ldots+x_k=j \\ x_1=M,\ x_i\leq M-1\ \forall i\in\{2,\ldots,k\}}} \cdots \sum \binom{M-1+j}{M-1,x_2,\ldots,x_k} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k} \\ & + \ldots + \sum_{\substack{x_1+\ldots+x_{k-1}=j \\ x_k=M,\ x_i\leq M-1\ \forall i\in\{1,\ldots,k-1\}}} \cdots \sum \binom{M-1+j}{M-1,x_1,\ldots,x_{k-1}} p_1^{x_1} \ldots p_{k-1}^{x_{k-1}} p_k^{x_k} \\ & 0 \leq j \leq (k-1)(M-1) \end{aligned}$$

and

$$E(\omega) \quad = \quad \sum_{j=0}^{(k-1)(M-1)} (M+j) Pr(\omega = M+j).$$

The corresponding calculations for Procedure 2 were done using simulation in Panchapakesan et al. (1998).

3 Homogeneity Testing for the Multivariate Hypergeometric Distribution

3.1 Introduction

Suppose that $X_1, X_2, ..., X_k$ are multivariate hypergeometric random variables with category sizes $M_1, M_2, ..., M_k$, where $M_1 + M_2 + ... + M_k = N$. The joint mass function of $X_1, X_2, ..., X_k$ is given by

$$P(X_1 = x_1, ..., X_k = x_k) = \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}...\binom{M_k}{x_k}}{\binom{N}{n}},$$

where $n = x_1 + x_2 + ... + x_k$ and $x_i \leq M_i$ for $i = 1, 2, ..., k$.

If M_i is big enough then

$$\binom{M_i}{x_i} = \frac{M_i!}{x_i!(M_i - x_i)!} = \frac{M_i(M_i - 1)...(M_i - x_i + 1)}{x_i!} \approx \frac{M_i^{x_i}}{x_i!},\ i = 1, 2, ..., k.$$

Therefore,

$$\begin{aligned} P(X_1 = x_1, ..., X_k = x_k) = & \quad \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}...\binom{M_k}{x_k}}{\binom{N}{n}} \approx \frac{\frac{M_1^{x_1} M_2^{x_2} ... M_k^{x_k}}{x_1! x_2! ... x_k!}}{\frac{N^n}{n!}} \\ = & \quad \binom{n}{x_1, x_2, ..., x_k}\left(\frac{M_1}{N}\right)^{x_1}\left(\frac{M_2}{N}\right)^{x_2}...\left(\frac{M_k}{N}\right)^{x_k}, \end{aligned}$$

where

$$\sum_{i=1}^{k} x_i = n$$

and

$$\sum_{i=1}^{k} \frac{M_i}{N} = 1.$$

So it is clear that the multivariate hypergeometric distribution approximate the multinomial distribution when N and M_i are large enough. That means that the multinomial distribution

is a special case of the multivariate hypergeometric distribution. We can continue to explore the relationship between the multinomial and the multivariate hypergeometric distributions.

For the multivariate hypergeometric distribution,

$$P(X_1 = x_1, ..., X_k = x_k) = \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}...\binom{M_k}{x_k}}{\binom{N}{n}} = \frac{n!}{x_1!x_2!...x_k!}\frac{\frac{M_1!}{(M_1-x_1)!}\frac{M_2!}{(M_2-x_2)!}\cdots\frac{M_k!}{(M_k-x_k)!}}{\frac{N!}{(N-n)!}}$$

$$= \binom{n}{x_1, x_2, ..., x_k}\{e^{\frac{1}{x_1}log\frac{\frac{M_1!}{M_1-x_1}}{(\frac{N!}{(N-n)!})^{1/k}}}\}^{x_1}\{e^{\frac{1}{x_2}log\frac{\frac{M_2!}{M_2-x_2}}{(\frac{N!}{(N-n)!})^{1/k}}}\}^{x_2}...\{e^{\frac{1}{x_k}log\frac{\frac{M_k!}{M_k-x_k}}{(\frac{N!}{(N-n)!})^{1/k}}}\}^{x_k}$$

If we let

$$p_i = e^{\frac{1}{x_i}log\frac{\frac{M_i!}{M_i-x_i}}{(\frac{N!}{(N-n)!})^{1/i}}}, \quad i = 1, 2, ..., k,$$

then

$$P(X_1 = x_1, ..., X_k = x_k) = \binom{n}{x_1, x_2, ..., x_k}p_1^{x_1}p_2^{x_2}...p_k^{x_k}.$$

But $p_1+p_2+...+p_k \neq 1$, and $p_1, p_2, ..., p_k$ are not constant, they are functions of $x_1, x_2, ..., x_k$, respectively.

In many situations, we have a finite population size N, for example when studying the animal population in a particular region. So the multivariate hypergeometric distribution provides a model for studying the diversity within a finite population which is classified into several categories based on some characteristic.

3.2 Procedure 1

The homogeneity null hypothesis for the multivariate hypergeometric distribution is as follows:

$$H_0 : M_1 = M_2 = ... = M_k.$$

The alternative hypothesis considered in this thesis is $H_a : M_1 = M_2 = ... = M_{k-1}, M_k = \lambda M_1$.

The proposed inverse sampling procedures are Procedures 1 and 2 described in Section 2.2. The critical values D_1 for Procedure 1 are obtained by finding the smallest integer among $\{1, 2, ..., M-1\}$ that satisfies $P_{H_0}(W_k \geq D_1) \leq \alpha$. And since $W_k = M - X_{(1)}$, D_1 must satisfy $X_{(1)} \leq M - D_1$. Suppose that $x_k = M$ at the termination time. Then at least one of the x_is $(i = 1, 2, ..., k-1)$ is less than $M - D_1$, or $\exists x_i$ such that $x_i \leq M - D_1$. So this event can be described as $\cup_{i=1}^{k-1}(x_i \leq M - D_1)$. Since the probabilities of $x_i = M$ at the termination time are the same for all $i = 1, 2, ..., k$, we have that

$$\begin{aligned} P_{H_0}(W_k \geq D_1) = & \quad kP_{H_0}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1), \cup_{i=1}^{k-1}(x_i \leq M - D_1)) \\ = & \quad kP_{H_0}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1)) \\ & -kP_{H_0}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1), \cap_{i=1}^{k-1}(x_i \geq M - D_1 + 1)) \end{aligned}$$

The probability of the event that one of the categories reaches M observations is 1 because the event must happen. And since each category is equally likely under H_0, at termination time we have that

$$P_{H_0}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1)) = \frac{1}{k}.$$

Therefore,

$$kP_{H_0}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1)) = 1$$

Just prior to category k reaching M observations, it will have $M-1$ observations. Suppose that category i has x_i observations at this time where $M - D_1 + 1 \leq x_i \leq min(M-1, M_i)$ $(i = 1, 2, ..., k-1)$. The probability of this event is

$$\frac{\binom{M_1}{x_1} \cdots \binom{M_{k-1}}{x_{k-1}} \binom{M_k}{M-1}}{\binom{N}{M-1+x_1+...+x_{k-1}}}.$$

Then since there are $M_k - M + 1$ observations remaining in category k and $N - M + 1 -$

$x_1 - ... - x_{k-1}$ remaining items in the population, the probability that the next observation comes from category k is $(M_k - M + 1)/(N - M + 1 - x_1 - ... - x_{k-1})$. Therefore,

$$P_{H_0}(W_k \geq D_1) = 1 - kP_{H_0}(x_k = M, \cap_{i=1}^{k-1}(M - D_1 + 1 \leq x_i \leq M - 1))$$

$$= 1 - k \sum_{x_{k-1}=M-D_1+1}^{min(M-1,M_{k-1})} ... \sum_{x_1=M-D_1+1}^{min(M-1,M_1)} \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}...\binom{M_{k-1}}{x_{k-1}}\binom{M_k}{M-1}}{\binom{N}{M-1+x_1+...+x_{k-1}}} \frac{M_k - M + 1}{N - M + 1 - x_1 - ... - x_{k-1}}.$$

Table 3.1: Critical values of D_1 for Procedure 1

(a) N = 60

k	α	M							
		8	9	10	11	12	13	14	15
3	0.01	8	9	9	10	10	10	11	11
	0.05	8	8	8	8	9	9	9	9
	0.10	7	7	8	8	8	8	8	8
	0.15	7	7	7	7	7	7	7	7
5	0.01	-	9	9	9	9	-	-	-
	0.05	8	8	8	8	7	-	-	-
	0.10	7	7	7	7	6	-	-	-
	0.15	7	7	7	7	6	-	-	-
6	0.01	8	9	8	-	-	-	-	-
	0.05	8	8	7	-	-	-	-	-
	0.10	7	7	7	-	-	-	-	-
	0.15	7	7	6	-	-	-	-	-

(b) N = 420

k	α	M							
		8	9	10	11	12	13	14	15
3	0.01	-	9	10	11	11	12	12	13
	0.05	8	9	9	10	10	10	11	11
	0.10	7	8	8	9	9	10	10	10
	0.15	7	7	8	8	9	9	9	10
5	0.01	-	-	-	11	12	13	13	14
	0.05	8	9	10	10	11	11	12	12
	0.10	8	9	9	10	10	11	11	11
	0.15	8	8	9	9	10	10	10	11
7	0.01	-	-	-	-	12	13	13	14
	0.05	-	9	10	11	11	12	12	13
	0.10	8	9	10	10	11	11	12	12
	0.15	8	9	9	10	10	11	11	11

Since the null hypothesis H_0 is $M_1 = M_2 = ... = M_k$, and $M_i = \frac{N}{k}$ should be an integer, and normally $k = 3(2)7, M = 8(1)15$, we chose $N = 60, 420, 1680, 1680000$ in our calculations. In Table 3.1, we present the critical values of D_1, where a missing entry "$-$" indicates that the given significance level was not attainable. For $N = 60$ we just use $k \leq 6$ since $\frac{60}{7}$ is not an integer and then for $k = 7$ the distribution is not a multivariate hypergeometric distribution.

From Table 3.1, we can see that the smaller N is, the smaller D_1 is. And, as the Ns get bigger, D_1 gradually stops changing. The results are the same as those in Table 1 of Pan-

Table 3.1: Critical values D_1 for Procedure 1

(c) N = 1680

k	α	M							
		8	9	10	11	12	13	14	15
3	0.01	-	-	10	11	11	12	13	13
	0.05	8	9	9	10	10	11	11	11
	0.10	7	8	8	9	9	10	10	10
	0.15	7	8	8	8	9	9	9	10
5	0.01	-	-	-	11	12	13	13	14
	0.05	-	9	10	10	11	12	12	13
	0.10	8	9	9	10	10	11	11	12
	0.15	8	8	9	9	10	10	11	11
7	0.01	-	-	-	-	12	13	14	14
	0.05	-	-	10	11	11	12	13	13
	0.10	8	9	10	10	11	11	12	12
	0.15	8	9	9	10	10	11	11	12

(d) N = 1680000

k	α	M							
		8	9	10	11	12	13	14	15
3	0.01	-	-	10	11	11	12	13	13
	0.05	8	9	9	10	10	11	11	12
	0.10	7	8	8	9	9	10	10	11
	0.15	7	8	8	8	9	9	10	10
5	0.01	-	-	-	11	12	13	13	14
	0.05	-	9	10	10	11	12	12	13
	0.10	8	9	9	10	10	11	11	12
	0.15	8	8	9	9	10	10	11	11
7	0.01	-	-	-	-	12	13	14	14
	0.05	-	-	10	11	12	12	13	13
	0.10	-	9	10	10	11	12	12	13
	0.15	8	9	9	10	11	11	12	12

chapakesan et al. (1998) when N is 1680000. Using this table and interpolation if needed, we can test homogeneity of the parameters of the multivariate hypergeometric distribution with typical N for a given population.

3.3 Procedure 2

For Procedure 2, in order to describe the exact calculation methods clearly we start with some definitions and theorems.

Definition 3.1: Let $X_{1t}, X_{2t}, ..., X_{kt}$ be the random variables at time t obtained using inverse sampling, and let $X_{(1)t} \leq X_{(2)t} \leq ... \leq X_{(k)t}$ be the order statistics in increasing order of magnitude. If $X_{1t}, X_{2t}, ..., X_{kt}$ satisfies the rejection criteria of Procedure 2, i.e., $X_{(k)t} - X_{(k-1)t} = D_2$, $(X_{1t}, X_{2t}, ..., X_{kt})$ is called a **target point**. If $X_{(k)t} - X_{(k-1)t} = D_2 - 1$, then $(X_{1t}, X_{2t}, ..., X_{kt})$ is called a **potential target point** in **layer** t, where D_2 can be any integer greater than 1.

Definition 3.2: We define the **target probability** at $(X_{1t}, X_{2t}, ..., X_{kt})$ to be the probability of $(X_{1t}, X_{2t}, ..., X_{kt})$ being a potential target point and then becoming a target point when the next observation is selected.

Theorem 3.3: Let $X_{1t} = x_1, X_{2t} = x_2, ..., X_{kt} = x_k$ be a random sample of size t from a multivariate hypergeometric distribution with $X_{(k)t} = X_{it}$ and $X_{(k-1)t} = X_{jt}$, where $i, j \in \{1, 2, ..., k\}$. The target probability at $(X_{1t}, X_{2t}, ..., X_{kt})$ is given by

$$B_i(x_1, x_2, ..., x_k) = \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}...\binom{M_k}{x_k}}{\binom{N}{x_1+...+x_{k-1}+x_k}} \frac{M_i - x_i}{N - x_1 - ... - x_{k-1} - x_k},$$

where $x_1 + x_2 + ... + x_k = t$ and $x_i = x_j + D_2 - 1$.

Proof: See Appendix A.1.1.

Definition 3.4: We call C_{N_1} the **target probability** at layer N_1 if C_{N_1} is the summation of all the target probabilities whose potential target points are in layer N_1.

Theorem 3.5: Under the null hypothesis, the target probability in layer N_1 is given by

$$\begin{aligned} C_{N_1} = \; & k\{\sum_{x_{k-1}=0}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} ... \sum_{x_1=0}^{x_{k-1}-1} \\ & B_k(x_1, ..., x_{k-1}, x_k)\}, \\ & {\scriptstyle x_k = x_{k-1}+D_2-1} \\ & {\scriptstyle x_{k-1}=x_{k-2}=...=x_{k-j}} \\ & {\scriptstyle x_1+x_2+...+x_k=N_1} \end{aligned} \tag{3.1}$$

where $B_k(x_1, ..., x_{k-1}, x_k)$ is the target probability at $X_{1t} = x_1, X_{2t} = x_2, ..., X_{kt} = x_{k-1}+D_2-1$.

Proof: See Appendix A.1.2.

For example, suppose that $k = 2, M_1 = 5, M_2 = 5, D_2 = 2$ and $N_0 = 4$. First we calculate target probability in layer 1 ($N_1 = 1$). From $x_1 + x_2 = 1$ and $x_2 = x_1 + D_2 - 1$, we get

$x_1 + x_1 + D_2 - 1 = 1$, and therefore $x_1 = 0$. Then from (3.1),

$$C_1 = \ 2\{\frac{\binom{5}{0}\binom{5}{0+2-1}}{\binom{10}{0+2-1}}\frac{5-0-2+1}{10-0-2+1}\} = 2 \times \frac{5}{10} \times \frac{4}{9} = \frac{4}{9}.$$

In order to illustrate the calculation of C_1 without using (3.1) suppose that an item in the population corresponding to $X_1 = 1$ is denoted by A, and corresponding to $X_2 = 1$ is denoted by B. Then AA and BB are both target points since both of them satisfy the condition of Procedure 2: $X_1 - X_2 = 2$ and $X_2 - X_1 = 2$. The probability of AA is $\frac{5}{10}\frac{4}{9}$ and the probability of BB is $\frac{5}{10}\frac{4}{9}$. The target probability in layer 1 is the same as the probability of the event set $\{AA, BB\}$. Therefore $C_1 = P(AA) + P(BB) = \frac{4}{9}$.

Using the formula from Theorem 3.5 we can also calculate that $C_3 = \frac{5}{14}$. To calculate C_3 without using Theorem 3.5 we note that the events in layer 3 are $ABAA, ABBB, BABB, BAAA, AABA, BBAB$. The probability of all these events is $6\frac{5}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7} = \frac{5}{14}$.

From the preceding example we see that some of the events in layer 3 (i.e. $AABA$ and $BBAB$) would not occur under the Procedure 2 sampling rule (since sampling would be terminated after AA and BB). In order to take this into account when calculating probabilities for Procedure 2, we first have to define the concept of a transfer probability.

Definition 3.6: For Procedure 2, we take one observation at a time and each time calculate $W_{kt} = X_{(k)t} - X_{(k-1)t}$. At the first time when $W_{kt} = D_2$, the procedure is stopped and we say that a **target event** has occurred. Now suppose that we don't stop sampling at the first target event but continue taking one observation at a time and each time calculate W_{kt}. Then at some point $W_{kt} = D_2$ might happen again just after reaching another potential target point, in which case we get a second target event (this occurs in the above example with $AABA$ and $BBAB$). Suppose that the potential target point of random variables of the first event $(x_1, x_2, ...x_k)$ is in layer N_1, while the potential target point of random variables of the second

event $(x'_1, x'_2, ...x'_k)$ is in layer N_2. We call the first event a **target event in layer** N_1, and the second event a **target event in layer** N_2. This whole process is called a target event in layer N_1 **transferred into** layer N_2. The probability of all of the target events in layer N_1 transferred into another layer N_2 is called the **transfer probability from** N_1 **to** N_2, and is denoted by C_{N_1,N_2}. Similarly, suppose that when we get the second target event we don't stop the procedure but continue taking one observation at a time and each time calculate W_{kt}. Then we can again get a target event, this time in layer N_3. The probability of all of the target events transferred from layer N_1 into layer N_2 and then transferred into layer N_3 is called the **transfer probability from** N_1 **to** N_2 **to** N_3, and is denoted by C_{N_1,N_2,N_3}. In general, the a-level transfer probability $C_{N_1,N_2,\ldots,N_a}$ is defined similarly.

For the above example, the two target events $AABA$ and $BBAB$ are in layer 3, but AA and BB are target events in layer 1. So the target event $AABA$ in layer 3 is transferred from AA in layer 1, and similarly for $BBAB$. Therefore the transfer probability from 1 to 3 is $C_{1,3} = 2 \times \frac{5}{10}\frac{4}{9}\frac{5}{8}\frac{3}{7} = \frac{5}{42}$.

Theorem 3.7: Suppose that $N_i \geq 1, i = 1, 2, 3, \ldots$. The transfer probability from N_1 to $N_1 + 1 + N_2$ and the transfer probability from N_1 to $N_1 + 1 + N_2$ to $N_1 + N_2 + 2 + N_3$ are given by

$$
\begin{aligned}
C_{N_1,N_1+1+N_2} = \quad & k\{\sum_{x_{k-1}=0}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} \cdots \sum_{x_1=0}^{x_{k-1}-1} \\
& \substack{x_k=x_{k-1}+D_2-1 \\ x_{k-1}=x_{k-2}=...=x_{k-j} \\ x_1+x_2+...+x_k=N_1} \\
& B_k(x_1,...,x_{k-1},x_k) \sum_{x'_k=0}^{M_k-x^c_k} \sum_{x'_{k-1}=0}^{M_{k-1}-x_{k-1}} \cdots \sum_{x'_1=0}^{M_1-x_1} \\
& \qquad \substack{x^c_k=x_k+1 \\ x'_1+x'_2+...+x'_k=N_2 \\ x'_a(k)=x_a+x'_a=max\{x^c_k+x'_k,x_{k-1}+x'_{k-1},...,x_1+x'_1\} \\ x'_b(k-1)=x_b+x'_b=secondmax\{x^c_k+x'_k,x_{k-1}+x'_{k-1},...,x_1+x'_1\} \\ x'_a(k)-x'_b(k-1)=D_2-1} \\
& \frac{\binom{M_1-x_1}{x'_1}\binom{M_2-x_2}{x'_2}...\binom{M_k-x^c_k}{x'_k}}{\binom{N-x_1-...-x^c_k}{N_2}} \frac{M_a-x_a-x'_a}{N-x_1-...-x^c_k-x'_1-...-x'_k}\},
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
C_{N_1,N_1+1+N_2,N_1+N_2+2+N_3} = \; & k\{\sum_{\substack{x_{k-1}=0 \\ x_k=x_{k-1}+D_2-1 \\ x_{k-1}=x_{k-2}=...=x_{k-j} \\ x_1+x_2+...+x_k=N_1}}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} \cdots \sum_{x_1=0}^{x_{k-1}-1} \\
& B_k(x_1,...,x_{k-1},x_k) \sum_{\substack{x'_k=0 \\ x^c_k=x_k+1 \\ x'_1+x'_2+...+x'_k=N_2 \\ x'_a(k)=x_a+x'_a=max\{x^c_k+x'_k,x_{k-1}+x'_{k-1},...,x_1+x'_1\} \\ x'_b(k-1)=x_b+x'_b=secondmax\{x^c_k+x'_k,x_{k-1}+x'_{k-1},...,x_1+x'_1\} \\ x'_a(k)-x'_b(k-1)=D_2-1}}^{M_k-x^c_k} \sum_{x'_{k-1}=0}^{M_{k-1}-x_{k-1}} \cdots \sum_{x'_1=0}^{M_1-x_1} \\
& \frac{\binom{M_1-x_1}{x'_1}\binom{M_2-x_2}{x'_2}...\binom{M_k-x^c_k}{x'_k}}{\binom{N-x_1-...-x^c_k}{N_2}} \frac{M_a-x_a-x'_a}{N-x_1-...-x^c_k-x'_1-...-x'_k} \\
& \sum_{\substack{x''_k=0 \\ x'^c_a=x'_a+1 \\ x''_1+x''_2+...+x''_k=N_3 \\ x''_{a'}(k)=x_{a'}+x'_{a'}+x''_{a'}=max\{x^c_k+x'_k+x''_k,x_{k-1}+x'_{k-1}+x''_{k-1},...,x_1+x'_1+x''_1\} \\ x''_{b'}(k-1)=x_{b'}+x'_{b'}+x''_{b'}=secondmax\{x^c_k+x'_k+x''_k,x_{k-1}+x'_{k-1}+x''_{k-1},...,x_1+x'_1+x''_1\} \\ x''_{a'}(k)-x''_{b'}(k-1)=D_2-1}}^{M_k-x^c_k-x'_k} \cdots \sum_{x''_a=0}^{M_a-x_a-x'^c_a} \cdots \sum_{x''_1=0}^{M_1-x_1-x'_1} \\
& \frac{\binom{M_1-x_1-x'_1}{x''_1}\binom{M_2-x_2-x'_2}{x''_2}...\binom{M_k-x^c_k-x'_k}{x''_k}}{\binom{N-x_1-...-x^c_k-x'_1-...-x'^c_a-...-x'_k}{N_3}} \frac{M_{a'}-x_{a'}-x'_{a'}-x''_{a'}}{N-x_1-...-x^c_k-x'_1-...-x'_k-x''_1-...-x''_k}\}.
\end{aligned}
\tag{3.3}
$$

Proof: See Appendix A.1.3.

In order to calculate probabilities for Procedure 2 we cannot include target events transferred from one layer to another. In some sense a target event transferred from layer N_1 to layer N_2 is an intersection of events in layer N_1 and N_2. But it is not exactly an intersection. For example, AA and $AABA$ are not the same events, but they do intersect in some sense. Therefore we need a more general definition of intersection.

Definition 3.8: The set of all the target events whose potential target points are in layer N_1 is called the **target event set at layer** N_1, and is denoted by S_{N_1}. Note that the probability of S_{N_1} is the target probability at layer N_1, i.e., $Pr(S_{N_1}) = C_{N_1}$. The set of all the target events whose potential target points are in layer N_2 which are transferred from layer N_1 is called the **right intersection from layer** N_1 **to** N_2, and is denoted by $S_{N_1}\tilde{\cap}S_{N_2}$. So the probability of $S_{N_1}\tilde{\cap}S_{N_2}$ is the transfer probability from N_1 to N_2, i.e., $Pr(S_{N_1}\tilde{\cap}S_{N_2}) = C_{N_1,N_2}$. The set of all the target events whose potential target points are in layer N_3 which are transferred from layer N_2 and transferred from layer N_1 is called the **right intersection from layer** N_1 **to** N_2 **to** N_3, and is denoted by $S_{N_1}\tilde{\cap}S_{N_2}\tilde{\cap}S_{N_3}$. And $Pr(S_{N_1}\tilde{\cap}S_{N_2}\tilde{\cap}S_{N_3}) = C_{N_1,N_2,N_3}$.

For example, suppose that $k = 2, M_1 = 7, M_2 = 7, D_2 = 2$ and $N_0 = 6$. Then the target event set at layer 1 is $S_1 = \{AA, BB\}$ and the target event set at layer 3 is $S_3 = \{ABAA, ABBB, BABB, BAAA, AABA, BBAB\}$. The right intersection from layer 1 to 3 is $S_1\tilde{\cap}S_3 = \{AABA, BBAB\}$.

The set of all target events whose potential target points are in layer N_1 and N_2 except those in $S_{N_1}\tilde{\cap}S_{N_2}$ is called the **left union of layer** N_1 **and** N_2, and is denoted by $S_{N_1}\tilde{\cup}S_{N_2}$. So $S_{N_1}\tilde{\cup}S_{N_2} = \{x \mid (x \in S_{N_1} \text{ or } x \in S_{N_2}) \text{ and } x \notin S_{N_1}\tilde{\cap}S_{N_2}\} = (S_{N_1} \cup S_{N_2}) \setminus (S_{N_1}\tilde{\cap}S_{N_2})$. So the probability of $S_{N_1}\tilde{\cup}S_{N_2}$ is $Pr(S_{N_1}\tilde{\cup}S_{N_2}) = C_{N_1} + C_{N_2} - C_{N_1,N_2}$. The set of all the target events in $S_{N_1}\tilde{\cup}S_{N_2}$ together with those whose potential target points are in layer N_3 except those in $(S_{N_1}\tilde{\cup}S_{N_2})\tilde{\cap}S_{N_3}$ is called the **left union of layer** N_1**,** N_2 **and** N_3, and is denoted by $S_{N_1}\tilde{\cup}S_{N_2}\tilde{\cup}S_{N_3}$ or $\tilde{\cup}_{i=1}^{3}S_{N_i}$. And $Pr(S_{N_1}\tilde{\cup}S_{N_2}\tilde{\cup}S_{N_3}) = Pr(S_{N_1}\tilde{\cup}S_{N_2}) + Pr(S_{N_3}) - Pr((S_{N_1}\tilde{\cup}S_{N_2})\tilde{\cap}S_{N_3})$. Since $(S_{N_1}\tilde{\cup}S_{N_2})\tilde{\cap}S_{N_3} = [(S_{N_1} \cup S_{N_2}) \setminus (S_{N_1}\tilde{\cap}S_{N_2})]\tilde{\cap}S_{N_3} = ((S_{N_1}\tilde{\cap}S_{N_3}) \cup (S_{N_2}\tilde{\cap}S_{N_3})) \setminus S_{N_1}\tilde{\cap}S_{N_2}\tilde{\cap}S_{N_3}$, we have that $Pr((S_{N_1}\tilde{\cup}S_{N_2})\tilde{\cap}S_{N_3}) = C_{N_1,N_3} + C_{N_2,N_3} - C_{N_1,N_2,N_3}$. So $Pr(S_{N_1}\tilde{\cup}S_{N_2}\tilde{\cup}S_{N_3}) = C_{N_1} + C_{N_2} - C_{N_1,N_2} + C_{N_3} - (C_{N_1,N_3} + C_{N_2,N_3} - C_{N_1,N_2,N_3}) = C_{N_1} + C_{N_2} + C_{N_3} - C_{N_1,N_2} - C_{N_1,N_3} - C_{N_2,N_3} + C_{N_1,N_2,N_3}$.

In the above example $S_1\tilde{\cup}S_3 = \{AA, BB, ABAA, ABBB, BABB, BAAA\}$, $S_5 = \{AABABA, ABBBAB, ABAABA, ABABAA, BABABB, ABBABB, ABABBB, BAABAA, BABAAA, BBABAB, BABBAB, AAABBA, AABBAA, BAABBB, AABBBB, ABBAAA,$

$BBAAAA, BBBAAB, BAAABA, BBAABB\}$. $S_1\tilde{\cup}S_3\tilde{\cup}S_5 = \{AA, BB, ABAA, ABBB, BABB, BAAA, ABABAA, BABABB, ABBABB, ABABBB, BAABAA, BABAAA, BAABBB, ABBAAA\}$. $(S_1\tilde{\cup}S_3)\tilde{\cap}S_5 = \{AABABA, ABBBAB, ABAABA, BBABAB, BABBAB, AAABBA, AABBAA, AABBBB, BBAAAA, BBBAAB, BAAABA, BBAABB\}$.

From calculating the probabilities of the target events in $S_1\tilde{\cup}S_3\tilde{\cup}S_5$, we have that

$$Pr(S_1\tilde{\cup}S_3\tilde{\cup}S_5) = \frac{7}{14}\frac{6}{13} \times 2 + \frac{7}{14}\frac{7}{13}\frac{6}{12}\frac{5}{11} \times 4 + \frac{7}{14}\frac{7}{13}\frac{6}{12}\frac{6}{11}\frac{5}{10}\frac{4}{9} \times 8 = 0.8368298$$

Let's check the results using the formula of $Pr(S_1\tilde{\cup}S_3\tilde{\cup}S_5)$:

$$\begin{aligned}
Pr(S_1) = C_1 &= \frac{7}{14}\frac{6}{13} \times 2 = 0.4615385, \\
Pr(S_3) = C_3 &= \frac{7}{14}\frac{7}{13}\frac{6}{12}\frac{5}{11} \times 6 = 0.3671329, \\
Pr(S_5) = C_5 &= \frac{7}{14}\frac{7}{13}\frac{6}{12}\frac{6}{11}\frac{5}{10}\frac{4}{9} \times 20 = 0.3263403, \\
C_{1,3} &= \frac{7}{14}\frac{7}{13}\frac{6}{12}\frac{5}{11} \times 2 = 0.1223776, \\
C_{1,5} &= \frac{7}{14}\frac{7}{13}\frac{6}{12}\frac{6}{11}\frac{5}{10}\frac{4}{9} \times 8 = 0.1305361, \\
C_{3,5} &= \frac{7}{14}\frac{7}{13}\frac{6}{12}\frac{6}{11}\frac{5}{10}\frac{4}{9} \times 6 = 0.0979021,
\end{aligned}$$

and

$$C_{1,3,5} = \frac{7}{14}\frac{7}{13}\frac{6}{12}\frac{6}{11}\frac{5}{10}\frac{4}{9} \times 2 = 0.03263403.$$

Therefore we have that

$$Pr(S_1 \tilde{\cup} S_3 \tilde{\cup} S_5) = C_1 + C_3 + C_5 - C_{1,3} - C_{1,5} - C_{3,5} + C_{1,3,5} =$$

$$= 0.4615385 + 0.3671329 + 0.3263403 - 0.1223776 - 0.1305361$$

$$-0.0979021 + 0.03263403 = 0.8368299.$$

The following theorem generalizes the results in the above example.

Theorem 3.9: The exact value of $P_{H_0}(X_{(k)t} - X_{(k-1)t} \geq D_2)$ is given by

$$P_{H_0}(X_{(k)t} - X_{(k-1)t} \geq D_2) = \sum_{i=D_2-1}^{N_0-1} C_i - \sum_{\substack{D_2-1 \leq i_1 < i_2 < N_0 \\ i_2 > i_1+1}} C_{i_1,i_2}$$

$$+ \ldots + (-1)^{r+1} \sum_{\substack{D_2-1 \leq i_1 < i_2 < \ldots < i_r < N_0 \\ i_2 > i_1+1, i_3 > i_2+1, \ldots, i_r > i_{r-1}+1 \\ r \leq \frac{N_0-D_2}{2}+1}} C_{i_1,i_2,\ldots,i_r}$$

Proof: See Appendix A.1.4.

3.4 Approximation Algorithm for $P_{H_0}(X_{(k)t} - X_{(k-1)t} \geq D_2)$

Since the algorithms for the transfer probabilities are complicated, a single computer can not complete the exact calculation of $P_{H_0}(X_{(k)t} - X_{(k-1)t} \geq D_2)$ in a reasonable amount of time for large k and $N_0 - D_2$. We can only use Theorem 3.9 to calculate the exact value of $P_{H_0}(X_{(k)t} - X_{(k-1)t} \geq D_2)$ when $N_0 - D_2$ and k are relatively small. In order to speed up the calculations we need to sacrifice some precision to quickly get the results. The lost precision should be acceptable and not affect the results and judgements. And the approximations should be comparable to the simulations and the exact results.

One possible solution is that we can just calculate the target probabilities C_{N_1} and the transfer probabilities with depth of 3, i.e., C_{N_1,N_1+1+N_2}'s and $C_{N_1,N_1+1+N_2,N_1+N_2+2+N_3}$'s since the transfer probabilities normally become smaller and smaller as the depths increase. But it already have needed heavy computation just with the depth 3.

We obtained another approximation whose results are very close to the exact results. Suppose we don't know transfer probabilities like C_{N_1,N_1+1+N_2}, $C_{N_1,N_1+1+N_2,N_1+N_2+2+N_3}$, but we just have target probabilities in layers like C_{N_1}. Suppose that we assume that target points in different layers are independent. Let

$$P_{N_1} = Pr(\tilde{\cup}_{i=D_2-1}^{N_1} S_i) = Pr((\tilde{\cup}_{i=D_2-1}^{N_1-1} S_i)\tilde{\cup} S_{N_1}) = Pr(\tilde{\cup}_{i=D_2-1}^{N_1-1} S_i) + Pr(S_{N_1}) \\ -Pr((\tilde{\cup}_{i=D_2-1}^{N_1-1} S_i)\tilde{\cap} S_{N_1}),$$

where S_i is the target event set at layer i for $i = D_2 - 1, D_2, ..., N_1$.

Suppose that

$$Pr((\tilde{\cup}_{i=D_2-1}^{N_1-1} S_i)\tilde{\cap} S_{N_1}) = Pr(\tilde{\cup}_{i=D_2-1}^{N_1-1} S_i) Pr(S_{N_1}).$$

Then we have that

$$P_{N_1} = P_{N_1-1} + C_{N_1} - P_{N_1-1} C_{N_1} = P_{N_1-1} + (1 - P_{N_1-1}) C_{N_1}.$$

Letting $N_1 = D_2 - 1, D_2, ..., N_0 - 1$, we get that

$$\begin{aligned} P_{D_2-1} &= C_{D_2-1} \\ P_{D_2} &= P_{D_2-1} + (1 - P_{D_2-1})C_{D_2} \\ ... & \quad ... \\ P_{N_0-1} &= P_{N_0-2} + (1 - P_{N_0-2})C_{N_0-1} \end{aligned}$$

Solving the above recursions for P_{N_0-1} gives

$$P_{H_0}(X_{(k)t} - X_{(k-1)t} \geq D_2) = P_{N_0-1} = \sum_{i=D_2-1}^{N_0-1} C_i - \sum_{D_2-1 \leq i_1 < i_2 < N_0} C_{i_1} C_{i_2}$$
$$+ ... + (-1)^{r+1} \sum_{D_2-1 \leq i_1 < i_2 < ... < i_r < N_0} C_{i_1} C_{i_2} ... C_{i_r} \qquad (3.4)$$

Note that the assumption $Pr((\tilde{\cup}_{i=D_2-1}^{N_1-1} S_i)\tilde{\cap} S_{N_1}) = Pr(\tilde{\cup}_{i=D_2-1}^{N_1-1} S_i) Pr(S_{N_1})$ is not correct. For example, there is no transfer probabilities from N_1 to $N_1 + 1$ according to Theorem 3.9. This means that $S_{N_1} \tilde{\cap} S_{N_1+1} = \emptyset$. But in this approximation method we assume that $Pr(S_{N_1} \tilde{\cap} S_{N_1+1}) = C_{N_1} C_{N_1+1}$. Although the assumption is incorrect, different layers are independent of each other in an average sense, so the above formula gives results that are very close to the exact results.

In Table 3.2, we present the exact critical values for D_2 obtained by Theorems 3.5, 3.7, and 3.9. The simulated critical values for D_2 and the approximate results obtained from (3.4) are shown in Tables 3.3 and 3.4, respectively. Comparing Table 3.2 with Table 3.3, we found 7 different points among a total 72 points (the ratio is $\frac{7}{72} = 9.7\%$). There are 22 different points between Tables 3.2 and 3.4 (the ratio is $\frac{22}{72} = 30.6\%$). So the simulated critical values are much closer to the exact critical values. We combined the exact critical values D_2 with the simulated critical values D_2 and revised them a little bit based on the approximate results. The resulting

Table 3.2: Exact Critical Values of D_2 for Procedure 2

(a) N = 60

k	α	N_0			
		18	21	24	27
3	0.01	8	8	8	9
	0.05	7	7	7	7
	0.1	6	6	6	7
	0.15	5	6	6	6

k	α	N_0	
		22	26
4	0.01	7	7
	0.05	6	6
	0.1	5	5
	0.15	5	5

(b) N = 420

k	α	N_0			
		18	21	24	27
3	0.01	9	10	10	11
	0.05	7	8	8	9
	0.1	6	7	7	8
	0.15	6	6	7	7

k	α	N_0	
		22	26
4	0.01	8	9
	0.05	7	7
	0.1	6	6
	0.15	5	6

(c) N = 1680, N = 1680000

k	α	N_0			
		18	21	24	27
3	0.01	9	10	10	11
	0.05	7	8	8	9
	0.1	7	7	7	8
	0.15	6	6	7	7

k	α	N_0	
		22	26
4	0.01	8	9
	0.05	7	7
	0.1	6	6
	0.15	5	6

critical values for D_2 are shown in Table 3.5.

From Table 3.5, we can see that the smaller N is, the smaller D_2 is under the same values of k, α, and N_0. And, as the N's get bigger, the D_2's gradually converge to stable values which agree with the values in Table 7 of Panchapakesan et al. (1998). Using this table, and interpolation if needed, we can test the homogeneity of the parameters of the multivariate hypergeometric distribution with typical N for a given population.

3.5 Simulation of Multivariate Hypergeometric Random Variables

Since C language has not a built-in routine for simulating multivariate hypergeometric random variables, we developed the following algorithm.

Suppose that $X_1, X_2, ..., X_k$ are multivariate hypergeometric random variables with category sizes $M_1, M_2, ..., M_k$, where $M_1 + M_2 + ... + M_k = N$. Let the current values of $X_1, X_2, ..., X_k$ be denoted by $x_1, x_2, ..., x_k$, where $x_1 \leq M_1$, $x_2 \leq M_2$, ..., $x_k \leq M_k$. We denote the next values of $X_1, X_2, ..., X_k$ by $x'_1, x'_2, ..., x'_k$. Let $s = x_1 + x_2 + ... + x_k$. To obtain $x'_1, x'_2, ..., x'_k$, we first produce a random number $I_x = Ceiling(Unif(0, N - s))$. Let $I'_x = I_x - M_1 + x_1$. If $I'_x \leq 0$, then we set $x'_1 = x_1 + 1$, $x'_2 = x_2, ...,$ and $x'_k = x_k$. If $I'_x > 0$, Let $I'_x = I_x - M_1 + x_1 - M_2 + x_2$.

Table 3.3: Simulated Critical Values of D_2 for Procedure 2

(a) N = 60

k	α	N_0			
		18	21	24	27
3	0.01	8	8	9	8
	0.05	7	7	7	7
	0.1	6	6	6	7
	0.15	5	6	6	6

k	α	N_0	
		22	26
4	0.01	7	7
	0.05	6	6
	0.1	5	5
	0.15	5	5

(b) N = 420

k	α	N_0			
		18	21	24	27
3	0.01	9	10	10	11
	0.05	7	8	8	8
	0.1	6	7	7	8
	0.15	6	6	7	7

k	α	N_0	
		22	26
4	0.01	8	9
	0.05	7	7
	0.1	6	6
	0.15	5	6

(c) N = 1680000

k	α	N_0			
		18	21	24	27
3	0.01	9	10	11	11
	0.05	7	8	8	9
	0.1	7	7	8	8
	0.15	6	7	7	7

k	α	N_0	
		22	26
4	0.01	8	9
	0.05	7	7
	0.1	6	6
	0.15	6	6

If $I'_x \leq 0$, then we set $x'_2 = x_2 + 1$, $x'_1 = x_1, x'_3 = x_3, ...,$ and $x'_k = x_k$. If $I'_x > 0$, Let $I'_x = I_x - M_1 + x_1 - M_2 + x_2 - M_3 + x_3$. The process is continued until $I'_x \leq 0$ at which point we get the values of $x'_1, x'_2, ..., x'_k$. Note that the process starts with $x_1 = x_2 = ... = x_k = 0$.

4 Powers of Procedures and Sample Size in Multivariate Hypergeometric Distribution

4.1 Powers of Procedure 1

To compute the power of Procedure 1, we consider the following alternative hypothesis:

$$H_a : M_1 = M_2 = ... = M_{k-1}, M_k = \lambda M_1.$$

Since $M_1 + M_2 + ... + M_k = N$, we must have that

$$M_1 = \frac{N}{k - 1 + \lambda}, M_k = \lambda M_1$$

Table 3.4: Approximate Critical Values of D_2 for Procedure 2

(a) N = 60

k	α	N_0			
		18	21	24	27
3	0.01	8	8	9	9
	0.05	7	7	7	8
	0.10	6	6	7	7
	0.15	6	6	6	6

k	α	N_0	
		22	26
4	0.01	7	7
	0.05	6	6
	0.10	5	6
	0.15	5	5

(b) N = 420

k	α	N_0			
		18	21	24	27
3	0.01	9	10	10	11
	0.05	8	8	9	9
	0.10	7	7	8	8
	0.15	6	7	7	7

k	α	N_0	
		22	26
4	0.01	8	9
	0.05	7	7
	0.10	6	7
	0.15	6	6

(c) N = 1680000

k	α	N_0			
		18	21	24	27
3	0.01	9	10	11	11
	0.05	8	8	9	9
	0.10	7	7	8	8
	0.15	6	7	7	8

k	α	N_0	
		22	26
4	0.01	9	9
	0.05	7	8
	0.10	6	7
	0.15	6	6

Therefore the power of Procedure 1 can be calculated as follows:

$$\begin{aligned}
P_{H_a}(W_k \geq D_1) &= (k-1)P_{H_a}(x_1 = M, \cap_{i=2}^{k}(x_i \leq M-1), \cup_{i=2}^{k}(x_i \leq M - D_1)) \\
&\quad + P_{H_a}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1), \cup_{i=1}^{k-1}(x_i \leq M - D_1)) \\
&= (k-1)[P_{H_a}(x_1 = M, \cap_{i=2}^{k}(x_i \leq M-1)) \\
&\quad - P_{H_a}(x_1 = M, \cap_{i=2}^{k}(x_i \leq M-1), \cap_{i=2}^{k}(x_i \geq M - D_1 + 1))] \\
&+ P_{H_a}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1)) - P_{H_a}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1), \cap_{i=1}^{k-1}(x_i \geq M - D_1 + 1)).
\end{aligned}$$

The probability that one of the categories reaches M observations is 1 because the event must happen. This event happens with all kinds of possible values of $X_1, X_2, ..., X_k$. Since each of the categories $1, 2, ..., k-1$ is equally likely under H_a, we have that

$$(k-1)[P_{H_a}(x_1 = M, \cap_{i=2}^{k}(x_i \leq M-1))] + P_{H_a}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1)) = 1.$$

Table 3.5: Critical values of D_2 for Procedure 2

(a) N = 60

k	α	N_0						
		18	21	24	27	30	33	36
3	0.01	8	8	8	9	9	9	9
	0.05	7	7	7	7	8	8	8
	0.1	6	6	6	7	7	7	7
	0.15	5	6	6	6	6	6	7
		N_0						
		25	30	35	40	45	50	55
5	0.01	6	6	6	6	6	6	6
	0.05	5	5	5	5	5	5	6
	0.1	5	5	5	5	5	5	5
	0.15	4	4	5	5	5	5	5
		N_0						
		29	35	41	47	53	59	65
6	0.01	5	6	6	6	6	6	-
	0.05	5	5	5	5	5	5	-
	0.1	4	4	4	4	4	4	-
	0.15	4	4	4	4	4	4	-

(b) N = 420

k	α	N_0						
		18	21	24	27	30	33	36
3	0.01	9	10	10	11	11	12	12
	0.05	7	8	8	9	9	9	10
	0.1	6	7	7	8	8	8	8
	0.15	6	6	7	7	7	8	8
		N_0						
		25	30	35	40	45	50	55
5	0.01	8	8	9	9	10	10	10
	0.05	6	6	7	7	8	8	8
	0.1	5	6	6	6	7	7	7
	0.15	5	5	6	6	6	6	7
		N_0						
		33	40	47	54	61	68	75
7	0.01	7	8	8	8	9	9	9
	0.05	6	6	6	7	7	7	8
	0.1	5	5	6	6	6	7	7
	0.15	5	5	5	5	6	6	6

Therefore

$$P_{H_a}(W_k \geq D_1) = 1 - (k-1)P_{H_a}(x_1 = M, \cap_{i=2}^{k}(x_i \leq M-1), \cap_{i=2}^{k}(x_i \geq M - D_1 + 1))$$
$$-P_{H_a}(x_k = M, \cap_{i=1}^{k-1}(x_i \leq M-1), \cap_{i=1}^{k-1}(x_i \geq M - D_1 + 1)).$$

Since $M_1 \neq M_k$ under the alternative hypothesis H_a, we consider that M observations can be obtained from either category 1 or category k. If we get at M observations from category k, then just prior to stopping time, category k will have $M-1$ observations and for the second P_{H_a} term in the above formula category i will have x_i observations satisfying $M - D_1 + 1 \leq x_i \leq min(M_i, M-1)$ for $i = 1, 2, ..., k-1$. For category 1, we have the similar

Table 3.5: Critical values of D_2 for Procedure 2

(c) N = 1680

k	α	N_0						
		18	21	24	27	30	33	36
3	0.01	9	10	10	11	11	12	12
	0.05	7	8	8	9	9	10	10
	0.1	7	7	7	8	8	9	9
	0.15	6	6	7	7	8	8	8
		N_0						
		25	30	35	40	45	50	55
5	0.01	8	8	9	9	10	11	11
	0.05	6	7	7	8	8	8	8
	0.1	6	6	6	7	7	7	8
	0.15	5	5	6	6	6	7	7
		N_0						
		33	40	47	54	61	68	75
7	0.01	7	8	8	9	9	10	10
	0.05	6	6	7	7	7	8	8
	0.1	5	6	6	6	7	7	7
	0.15	5	5	5	6	6	6	7

(d) N = 1680000

k	α	N_0						
		18	21	24	27	30	33	36
3	0.01	9	10	10	11	11	12	12
	0.05	7	8	8	9	9	10	10
	0.1	7	7	7	8	8	8	9
	0.15	6	6	7	7	8	8	8
		N_0						
		25	30	35	40	45	50	55
5	0.01	8	8	9	9	10	11	11
	0.05	6	7	7	8	8	9	9
	0.1	5	6	6	7	7	7	7
	0.15	5	5	6	6	6	7	7
		N_0						
		33	40	47	54	61	68	75
7	0.01	8	8	9	9	10	10	10
	0.05	6	6	7	7	8	8	8
	0.1	5	6	6	6	7	7	7
	0.15	5	5	6	6	6	6	7

results. Therefore,

$$P_{H_a}(W_k \geq D_1) =$$

$$1-(k-1)\sum_{\substack{x_2=M-D_1+1 \\ M\leq M_1}}^{min(M_2,M-1)} \cdots \sum_{x_k=M-D_1+1}^{min(M_k,M-1)} \frac{\binom{M_1}{M-1}\binom{M_2}{x_2}\cdots\binom{M_k}{x_k}}{\binom{N}{M-1+x_2+\ldots+x_k}} \frac{M_1-M+1}{N-M+1-x_2-\ldots-x_k}$$

$$-\sum_{x_1=M-D_1+1}^{min(M_1,M-1)} \cdots \sum_{\substack{x_{k-1}=M-D_1+1 \\ M\leq M_k}}^{min(M_{k-1},M-1)} \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}\cdots\binom{M_{k-1}}{x_{k-1}}\binom{M_k}{M-1}}{\binom{N}{M-1+x_1+\ldots+x_{k-1}}} \frac{M_k-M+1}{N-M+1-x_1-\ldots-x_{k-1}}.$$

In Tables 4.1 and 4.2, we present the power of Procedure 1 for values of λ as close as possible to $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 2, 3, 4, 5, 1$, $k = 3(2)7$, $M = 8(1)15$, and $\alpha = 0.05$ and 0.10, respectively, where a

missing entry in Tables 4.1 and 4.2 indicates that D_1 does not exist. For $N = 60$, the powers increase more quickly as M increases compared with those for $N = 420, 1680$, and 1680000 under the same k and λ. Comparing the powers of Procedure 1 when $N = 420, 1680$, and 1680000 we found that the results of $N = 420$ are a little bit bigger than those of $N = 1680$ and 1680000 under the same k, λ, and M. When $\lambda < 1$, the smaller λ is, the bigger the powers of Procedure 1 are for all values of k and M. When $\lambda > 1$, the bigger λ is, the bigger the powers of Procedure 1 are for all values of k and M. For example, looking at the fifth row of $k = 5$ under $N = 420$ when $\lambda = 2$ in Table 4.1 we can see that the power is 0.1222, while in the same row when $\lambda = 4$, the power is 0.5708. When $N \geq 1680000$ the results of Tables 4.1 and 4.2 are the same as Tables 2 and 3 in the paper of Panchapakesan et al. (1998).

But Procedure 1 may not be the final procedure, we need to study more powerful procedures. If we use a procedure which is a kind of procedure combined with Procedure 1 and exact calculations of the powers of this procedure require heavy computational effort and time, we can not give the exact powers of the procedure. Can we use simulations as a replacement of the exact calculations? Therefore we did the simulated powers of Procedure 1 in Table 4.3 at the 5% significance level. The results of Table 4.3 are very close to those of Table 4.1 for all values of k, M, and λ. Therefore simulations can be used in the power calculations of Procedure 1. Since the sample size or waiting time ω is not fixed for Procedure 1, we also computed the expected sample size for Procedure 1. The expected sample size $E(\omega)$ under H_a can be obtained by the following formula:

$$
\begin{aligned}
Pr(\omega = M + j) = & \\
& (k-1) \sum \cdots \sum_{\substack{x_2+\ldots+x_k=j \\ x_i \leq min(M_i, M-1), i \neq 1}} \frac{\binom{M_1}{M-1}\binom{M_1}{x_2}\cdots\binom{M_1}{x_{k-1}}\binom{M_k}{x_k}}{\binom{N}{M-1+x_2+\ldots+x_k}} \frac{M_1-M+1}{N-M+1-x_2-\ldots-x_k} \\
+ & \sum \cdots \sum_{\substack{x_1+\ldots+x_{k-1}=j \\ x_i \leq min(M_i, M-1), i \neq k}} \frac{\binom{M_1}{x_1}\binom{M_1}{x_2}\cdots\binom{M_1}{x_{k-1}}\binom{M_k}{M-1}}{\binom{N}{M-1+x_1+\ldots+x_{k-1}}} \frac{M_k-M+1}{N-M+1-x_1-\ldots-x_{k-1}} \\
& 0 \leq j \leq (k-1)(M-1),
\end{aligned}
$$

and

$$E(\omega) \;=\; \sum_{j=0}^{(k-1)(M-1)}(M+j)Pr(\omega = M+j).$$

In Table 4.4, we present the expected sample size for Procedure 1, where a missing entry indicates that the critical value D_1 does not exist. The simulated sample size for Procedure 1 is shown in Table 4.5. When $\lambda < 1$, under $N = 60$ when $k \geq 5$, the expected sample sizes are bigger and increase more quickly as M increases compared with those under $N = 420, 1680$, and 1680000 for all values of k and λ. For example, looking at the column of $\lambda = 0.5$ of $k = 5$, under $N = 60$ in Table 4.4 when $M = 8$, the expected sample size is 26.67 and when $M = 13$, the expected sample size is 52.18 while under $N = 1680$, when $M = 8$, the expected sample size is 15.95 and when $M = 13$, the expected sample size is 32.08. When $\lambda > 1$, the expected sample sizes of Procedure 1 under all $N = 60, 420, 1680$, and 1680000 are generally very close for all values of k, λ, and M. And the expected sample sizes of Procedure 1 are almost the same as the simulated results for all values of k, M, and λ. For all values of k, M, and N, the expected sample sizes are the biggest when $\lambda = 1$. When $\lambda < 1$, the smaller λ is, the smaller the expected sample sizes are. When $\lambda > 1$, the bigger λ is, the smaller the expected sample sizes are. When $N \geq 1680000$ the results of Tables 4.4 and 4.5 are same as Tables 5 and 6 in the paper of Panchapakesan et al. (1998).

Table 4.1: Exact powers of Procedure 1 at the 5% significance level

(a) N = 60

k	M	λ								D1
		0.222	0.308	0.5	2	3	4	4.67	1	
3	8	0.2182	0.1281	0.0432	0.0528	0.1487	0.2564	0.3246	0.0091	8
	9	0.4938	0.3187	0.1164	0.1410	0.3576	0.5484	0.6462	0.0227	8
	10	0.7245	0.5154	0.2055	0.2456	0.5577	0.7650	0.8477	0.0368	8
	11	0.8780	0.6873	0.3018	0.3546	0.7194	0.8941	0.9457	0.0491	8
	12	0.8370	0.6096	0.2183	0.2688	0.6445	0.8547	0.9225	0.0216	9
	13	0.9420	0.7615	0.3042	0.3672	0.7797	0.9399	0.9760	0.0259	9
	14	0.9883	0.8760	0.3950	0.4659	0.8759	0.9795	0.9943	0.0281	9
	15	1.0000	0.9501	0.4886	0.5616	0.9377	0.9945	0.9990	0.0281	9

k	M	λ								D1
		0.286	0.286	0.615	2	2.67	3.5	4.57	1	
5	8	0.1501	0.1501	0.0273	0.0786	0.1792	0.3210	0.4853	0.0117	8
	9	0.3763	0.3763	0.0651	0.1987	0.4142	0.6418	0.8212	0.0233	8
	10	0.6197	0.6197	0.1015	0.3313	0.6255	0.8461	0.9554	0.0278	8
	11	0.8418	0.8418	0.1296	0.4628	0.7860	0.9471	0.9925	0.0233	8
	12	1.0000	1.0000	0.3326	0.8046	0.9772	0.9990	1.0000	0.0367	7

k	M	λ								D1
		0	0.455	0.455	1.67	2.5	3.57	5	1	
6	8	1.0000	0.0491	0.0491	0.0418	0.1676	0.3741	0.6005	0.0094	8
	9	1.0000	0.1187	0.1187	0.1020	0.3903	0.7086	0.9043	0.0141	8
	10	1.0000	0.4754	0.4754	0.3739	0.8496	0.9857	0.9995	0.0366	7

(b) N = 420

k	M	λ									D1
		0.199	0.246	0.333	0.5	2	3	4	5	1	
3	8	0.2971	0.2282	0.1432	0.0645	0.0734	0.1793	0.2913	0.3920	0.0190	8
	9	0.2482	0.1831	0.1068	0.0420	0.0485	0.1356	0.2367	0.3331	0.0094	9
	10	0.5221	0.4156	0.2681	0.1171	0.1342	0.3307	0.5105	0.6466	0.0273	9
	11	0.4660	0.3579	0.2159	0.0830	0.0963	0.2704	0.4465	0.5887	0.0146	10
	12	0.6766	0.5566	0.3694	0.1576	0.1815	0.4486	0.6605	0.7952	0.0290	10
	13	0.8181	0.7119	0.5139	0.2416	0.2758	0.6042	0.8067	0.9070	0.0466	10
	14	0.7814	0.6624	0.4523	0.1894	0.2189	0.5410	0.7628	0.8810	0.0275	11
	15	0.8792	0.7828	0.5786	0.2670	0.3058	0.6713	0.8663	0.9470	0.0406	11

k	M	λ									D1
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1	
5	8	0.3648	0.3002	0.2057	0.1044	0.1381	0.3069	0.4583	0.5923	0.0495	8
	9	0.3029	0.2396	0.1516	0.0654	0.0914	0.2377	0.3826	0.5201	0.0249	9
	10	0.2520	0.1918	0.1120	0.0410	0.0602	0.1824	0.3163	0.4524	0.0123	10
	11	0.5398	0.4421	0.2884	0.1193	0.1703	0.4318	0.6383	0.7881	0.0373	10
	12	0.4813	0.3809	0.2310	0.0815	0.1222	0.3600	0.5708	0.7365	0.0197	11
	13	0.7017	0.5936	0.3999	0.1595	0.2318	0.5727	0.7880	0.9074	0.0403	11
	14	0.6519	0.5352	0.3371	0.1153	0.1758	0.5021	0.7362	0.8780	0.0222	12
	15	0.8093	0.7051	0.4908	0.1899	0.2801	0.6747	0.8749	0.9596	0.0381	12

k	M	λ									D1
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1	
7	8	-	-	-	-	-	-	-	-	-	-
	9	0.3794	0.2402	0.1569	0.1070	0.1178	0.3045	0.5040	0.6568	0.0427	9
	10	0.3199	0.1850	0.1101	0.0684	0.0758	0.2353	0.4272	0.5870	0.0210	10
	11	0.2703	0.1430	0.0775	0.0438	0.0484	0.1798	0.3579	0.5191	0.0100	11
	12	0.5796	0.3626	0.2190	0.1325	0.1464	0.4388	0.7004	0.8510	0.0323	11
	13	0.5238	0.3025	0.1675	0.0923	0.1019	0.3649	0.6354	0.8077	0.0164	12
	14	0.7530	0.5078	0.3126	0.1847	0.2028	0.5876	0.8455	0.9479	0.0349	12
	15	0.7084	0.4450	0.2524	0.1362	0.1494	0.5149	0.8007	0.9277	0.0184	13

(c) N = 1680

k	M	λ								D1
		0.199	0.333	0.5	2	3	4	5	1	
3	8	0.3020	0.1472	0.0672	0.0761	0.1830	0.2954	0.3961	0.0204	8
	9	0.2535	0.1107	0.0444	0.0507	0.1393	0.2411	0.3377	0.0103	9
	10	0.5263	0.2744	0.1224	0.1393	0.3363	0.5149	0.6495	0.0301	9
	11	0.4713	0.2225	0.0878	0.1010	0.2764	0.4518	0.5926	0.0165	10
	12	0.6778	0.3765	0.1651	0.1886	0.4545	0.6633	0.7959	0.0329	10
	13	0.6301	0.3201	0.1250	0.1441	0.3917	0.6072	0.7534	0.0190	11
	14	0.7795	0.4590	0.1988	0.2277	0.5462	0.7637	0.8800	0.0322	11
	15	0.8747	0.5825	0.2778	0.3156	0.6739	0.8650	0.9450	0.0476	11

k	M	λ								D1
		0.2	0.33	0.504	2	3	4	4.98	1	
	9	0.3150	0.1617	0.0757	0.0983	0.2464	0.4026	0.5337	0.0291	9
	10	0.2646	0.1214	0.0491	0.0658	0.1909	0.3365	0.4673	0.0150	10
	11	0.5511	0.3046	0.1397	0.1832	0.4438	0.6601	0.7980	0.0455	10
	12	0.4947	0.2475	0.0989	0.1337	0.3731	0.5955	0.7489	0.0253	11
	13	0.4426	0.2005	0.0698	0.0967	0.3101	0.5319	0.6971	0.0139	12
	14	0.6595	0.3572	0.1416	0.1925	0.5163	0.7572	0.8847	0.0302	12
	15	0.6126	0.3033	0.1054	0.1467	0.4507	0.7060	0.8530	0.0173	13

(d) N = 1680000

k	M	λ									D1
		0.2	0.25	0.333	0.5	2	3	4	5	1	
3	8	0.3019	0.2295	0.1485	0.0681	0.0770	0.1843	0.2968	0.3975	0.0209	8
	9	0.2536	0.1850	0.1120	0.0451	0.0515	0.1405	0.2425	0.3392	0.0107	9
	10	0.5251	0.4148	0.2765	0.1241	0.1410	0.3381	0.5163	0.6504	0.0311	9
	11	0.4703	0.3584	0.2247	0.0894	0.1025	0.2784	0.4535	0.5939	0.0172	10
	12	0.6755	0.5523	0.3787	0.1675	0.1910	0.4564	0.6642	0.7960	0.0343	10
	13	0.6280	0.4974	0.3227	0.1273	0.1463	0.3940	0.6085	0.7539	0.0200	11
	14	0.7764	0.6542	0.4611	0.2019	0.2306	0.5479	0.7640	0.8796	0.0338	11
	15	0.7387	0.6054	0.4055	0.1592	0.1832	0.4882	0.7187	0.8508	0.0204	12

k	M	λ									D1
		0.2	0.25	0.333	0.5	2	3	4	5	1	
5	8	-	-	-	-	-	-	-	-	-	-
	9	0.3189	0.2453	0.1623	0.0793	0.1006	0.2493	0.4054	0.5382	0.0306	9
	10	0.2687	0.1981	0.1222	0.0520	0.0677	0.1937	0.3396	0.4722	0.0159	10
	11	0.5546	0.4452	0.3050	0.1468	0.1875	0.4478	0.6627	0.8011	0.0484	10
	12	0.4989	0.3861	0.2483	0.1050	0.1376	0.3774	0.5987	0.7528	0.0274	11
	13	0.4475	0.3339	0.2016	0.0749	0.1001	0.3147	0.5357	0.7019	0.0152	12
	14	0.6619	0.5343	0.3573	0.1506	0.1981	0.5208	0.7594	0.8868	0.0332	12
	15	0.6158	0.4814	0.3040	0.1134	0.1519	0.4559	0.7088	0.8558	0.0194	13

Table 4.2: Exact powers of Procedure 1 at the 10% significance level

(a) N = 60

k	M	λ								D1
		0.222	0.308	0.5	2	3	4	4.67	1	
3	8	0.5727	0.4024	0.1786	0.2074	0.4440	0.6281	0.7161	0.0512	7
	9	0.7859	0.6029	0.2909	0.3333	0.6432	0.8226	0.8888	0.0786	7
	10	0.7245	0.5154	0.2055	0.2456	0.5577	0.7650	0.8477	0.0368	8
	11	0.8780	0.6873	0.3018	0.3546	0.7194	0.8941	0.9457	0.0491	8
	12	0.9599	0.8208	0.4000	0.4613	0.8363	0.9593	0.9844	0.0584	8
	13	0.9925	0.9129	0.4973	0.5618	0.9132	0.9870	0.9966	0.0642	8
	14	1.0000	0.9674	0.5920	0.6539	0.9590	0.9968	0.9995	0.0661	8
	15	1.0000	0.9928	0.6828	0.7362	0.9834	0.9994	1.0000	0.0642	8

k	M	λ								D1
		0.286	0.286	0.615	2	2.67	3.5	4.57	1	
5	8	0.4955	0.4955	0.1410	0.3007	0.5261	0.7324	0.8764	0.0713	7
	9	0.7299	0.7299	0.2089	0.4551	0.7266	0.8991	0.9736	0.0903	7
	10	0.9042	0.9042	0.2641	0.5909	0.8589	0.9694	0.9961	0.0903	7
	11	1.0000	1.0000	0.3060	0.7072	0.9370	0.9931	0.9997	0.0713	7
	12	1.0000	1.0000	0.6075	0.9368	0.9975	1.0000	1.0000	0.0981	6
	13	-	-	-	-	-	-	-	-	-
	14	-	-	-	-	-	-	-	-	-
	15	-	-	-	-	-	-	-	-	-

k	M	λ								D1
		0	0.455	0.455	1.67	2.5	3.57	5	1	
6	8	1.0000	0.2359	0.2359	0.1933	0.5131	0.7945	0.9398	0.0617	7
	9	1.0000	0.3577	0.3577	0.2875	0.7127	0.9367	0.9926	0.0617	7
	10	1.0000	0.4754	0.4754	0.3739	0.8496	0.9857	0.9995	0.0366	7
	11	-	-	-	-	-	-	-	-	-
	12	-	-	-	-	-	-	-	-	-
	13	-	-	-	-	-	-	-	-	-
	14	-	-	-	-	-	-	-	-	-
	15	-	-	-	-	-	-	-	-	-

(b) N = 420

k	M	λ								D1
		0.199	0.246	0.333	0.5	2	3	4	5	1	
3	8	0.6458	0.5511	0.4055	0.2278	0.2527	0.4790	0.6489	0.7618	0.0899	7
	9	0.5822	0.4800	0.3309	0.1640	0.1852	0.4003	0.5784	0.7047	0.0501	8
	10	0.7734	0.6753	0.5049	0.2734	0.3057	0.5894	0.7714	0.8721	0.0867	8
	11	0.7254	0.6152	0.4338	0.2087	0.2372	0.5173	0.7170	0.8355	0.0508	9
	12	0.8525	0.7603	0.5788	0.3046	0.3427	0.6677	0.8469	0.9295	0.0772	9
	13	0.8181	0.7119	0.5139	0.2416	0.2758	0.6042	0.8067	0.9070	0.0466	10
	14	0.9032	0.8211	0.6371	0.3278	0.3704	0.7264	0.8956	0.9605	0.0660	10
	15	0.9507	0.8928	0.7357	0.4114	0.4600	0.8162	0.9459	0.9840	0.0865	10

k	M	λ								D1
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1	
5	8	0.3648	0.3002	0.2057	0.1044	0.1381	0.3069	0.4583	0.5923	0.0495	8
	9	0.3029	0.2396	0.1516	0.0654	0.0914	0.2377	0.3826	0.5201	0.0249	9
	10	0.6027	0.5105	0.3579	0.1731	0.2342	0.5108	0.7055	0.8357	0.0687	9
	11	0.5398	0.4421	0.2884	0.1193	0.1703	0.4318	0.6383	0.7881	0.0373	10
	12	0.7511	0.6540	0.4703	0.2182	0.3010	0.6444	0.8354	0.9323	0.0711	10
	13	0.7017	0.5936	0.3999	0.1595	0.2318	0.5727	0.7880	0.9074	0.0403	11
	14	0.8447	0.7541	0.5570	0.2494	0.3501	0.7355	0.9060	0.9716	0.0657	11
	15	0.9255	0.8607	0.6871	0.3427	0.4642	0.8451	0.9614	0.9920	0.0940	11

k	M	λ								D1
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1	
7	8	0.4513	0.3130	0.2241	0.1671	0.1811	0.3889	0.5869	0.7263	0.0842	8
	9	0.3794	0.2402	0.1569	0.1070	0.1178	0.3045	0.5040	0.6568	0.0427	9
	10	0.3199	0.1850	0.1101	0.0684	0.0758	0.2353	0.4272	0.5870	0.0210	10
	11	0.6387	0.4323	0.2848	0.1887	0.2071	0.5195	0.7626	0.8890	0.0615	10
	12	0.5796	0.3626	0.2190	0.1325	0.1464	0.4388	0.7004	0.8510	0.0323	11
	13	0.7962	0.5748	0.3836	0.2477	0.2705	0.6607	0.8846	0.9639	0.0642	11
	14	0.7530	0.5078	0.3126	0.1847	0.2028	0.5876	0.8455	0.9479	0.0349	12
	15	0.8891	0.6816	0.4627	0.2921	0.3171	0.7554	0.9430	0.9881	0.0586	12

(c) N = 1680

k	M	λ								D1
		0.199	0.333	0.5	2	3	4	5	1	
3	8	0.6479	0.4105	0.2335	0.2581	0.4830	0.6512	0.7630	0.0950	7
	9	0.5853	0.3366	0.1696	0.1906	0.4052	0.5818	0.7067	0.0540	8
	10	0.7728	0.5095	0.2808	0.3126	0.5929	0.7722	0.8717	0.0935	8
	11	0.7256	0.4397	0.2163	0.2443	0.5219	0.7187	0.8355	0.0560	9
	12	0.8496	0.5824	0.3134	0.3507	0.6702	0.8463	0.9280	0.0853	9
	13	0.8156	0.5191	0.2509	0.2843	0.6080	0.8067	0.9057	0.0529	10
	14	0.8989	0.6392	0.3378	0.3793	0.7277	0.8940	0.9587	0.0750	10
	15	0.9462	0.7348	0.4210	0.4683	0.8152	0.9436	0.9826	0.0985	10

k	M	λ								D1
		0.2	0.33	0.504	2	3	4	4.98	1	
5	8	0.3761	0.2162	0.1170	0.1461	0.3155	0.4773	0.6043	0.0556	8
	9	0.3150	0.1617	0.0757	0.0983	0.2464	0.4026	0.5337	0.0291	9
	10	0.6118	0.3733	0.1963	0.2481	0.5214	0.7239	0.8432	0.0801	9
	11	0.5511	0.3046	0.1397	0.1832	0.4438	0.6601	0.7980	0.0455	10
	12	0.7544	0.4866	0.2501	0.3191	0.6539	0.8489	0.9355	0.0868	10
	13	0.7069	0.4183	0.1890	0.2495	0.5846	0.8052	0.9122	0.0518	11
	14	0.8425	0.5719	0.2889	0.3712	0.7430	0.9147	0.9726	0.0846	11
	15	0.8081	0.5087	0.2273	0.3016	0.6849	0.8866	0.9613	0.0521	12

(d) N = 1680000

k	M	λ								D1
		0.2	0.25	0.333	0.5	2	3	4	5	1	
3	8	0.6464	0.5482	0.4121	0.2354	0.2599	0.4843	0.6520	0.7633	0.0967	7
	9	0.5839	0.4780	0.3385	0.1715	0.1924	0.4068	0.5829	0.7074	0.0553	8
	10	0.7705	0.6690	0.5110	0.2833	0.3148	0.5940	0.7725	0.8715	0.0958	8
	11	0.7232	0.6097	0.4416	0.2188	0.2467	0.5235	0.7193	0.8355	0.0578	9
	12	0.8469	0.7511	0.5836	0.3163	0.3533	0.6710	0.8460	0.9276	0.0880	9
	13	0.8126	0.7030	0.5207	0.2539	0.2871	0.6093	0.8067	0.9053	0.0550	10
	14	0.8960	0.8096	0.6400	0.3410	0.3822	0.7281	0.8934	0.9582	0.0781	10
	15	0.8714	0.7710	0.5837	0.2813	0.3187	0.6748	0.8645	0.9444	0.0500	11

k	M	λ								D1
		0.2	0.25	0.333	0.5	2	3	4	5	1	
5	8	0.3797	0.3050	0.2168	0.1213	0.1488	0.3184	0.4798	0.6082	0.0577	8
	9	0.3189	0.2453	0.1623	0.0793	0.1006	0.2493	0.4054	0.5382	0.0306	9
	10	0.6148	0.5118	0.3734	0.2041	0.2527	0.5248	0.7259	0.8456	0.0841	9
	11	0.5546	0.4452	0.3050	0.1468	0.1875	0.4478	0.6627	0.8011	0.0484	10
	12	0.7555	0.6488	0.4857	0.2607	0.3250	0.6570	0.8498	0.9366	0.0922	10
	13	0.7086	0.5903	0.4178	0.1989	0.2554	0.5884	0.8067	0.9138	0.0559	11
	14	0.8419	0.7425	0.5697	0.3018	0.3781	0.7454	0.9150	0.9730	0.0912	11
	15	0.8079	0.6945	0.5070	0.2398	0.3087	0.6881	0.8872	0.9619	0.0572	12

Table 4.3: Simulated powers of Procedure 1 at the 5% significance level

(a) N = 60

k	M	λ								D1
		0.222	0.308	0.5	2	3	4	4.67	1	
3	8	0.2198	0.1281	0.0432	0.0508	0.1460	0.2524	0.3261	0.0099	8
	9	0.4874	0.3203	0.1169	0.1419	0.3494	0.5444	0.6505	0.0217	8
	10	0.7224	0.5186	0.2102	0.2431	0.5560	0.7700	0.8465	0.0363	8
	11	0.8794	0.6941	0.3014	0.3571	0.7168	0.8940	0.9452	0.0516	8
	12	0.8337	0.6045	0.2198	0.2672	0.6444	0.8541	0.9225	0.0224	9
	13	0.9414	0.7610	0.3063	0.3676	0.7847	0.9412	0.9749	0.0277	9
	14	0.9877	0.8766	0.3983	0.4674	0.8754	0.9800	0.9948	0.0278	9
	15	0.9823	0.9504	0.4854	0.5614	0.9389	0.9946	0.9986	0.0261	9

k	M	λ								D1
		0.286	0.286	0.615	2	2.67	3.5	4.57	1	
5	8	0.1559	0.1576	0.0278	0.0747	0.1789	0.3229	0.4735	0.0115	8
	9	0.3780	0.3791	0.0669	0.1995	0.4093	0.6373	0.8172	0.0236	8
	10	0.6220	0.6196	0.0979	0.3296	0.6214	0.8460	0.9534	0.0270	8
	11	0.8439	0.8386	0.1259	0.4602	0.7845	0.9426	0.9889	0.0236	8
	12	0.7492	0.7490	0.3281	0.7843	0.9625	0.9886	0.9918	0.0346	7
	13	-	-	-	-	-	-	-	-	-
	14	-	-	-	-	-	-	-	-	-
	15	-	-	-	-	-	-	-	-	-

k	M	λ								D1
		0	0.455	0.455	1.67	2.5	3.57	5	1	
6	8	1.0000	0.0494	0.0493	0.0416	0.1691	0.3757	0.5971	0.0091	8
	9	1.0000	0.1184	0.1181	0.1044	0.3873	0.7067	0.9015	0.0125	8
	10	1.0000	0.4724	0.4730	0.3569	0.8290	0.9715	0.9844	0.0359	7
	11	-	-	-	-	-	-	-	-	-
	12	-	-	-	-	-	-	-	-	-
	13	-	-	-	-	-	-	-	-	-
	14	-	-	-	-	-	-	-	-	-
	15	-	-	-	-	-	-	-	-	-

(b) N = 420

k	M	λ									D1
		0.199	0.246	0.333	0.5	2	3	4	5	1	
3	8	0.2895	0.2211	0.1419	0.0628	0.0732	0.1768	0.2925	0.3916	0.0198	8
	9	0.2477	0.1871	0.1099	0.0441	0.0500	0.1365	0.2357	0.3303	0.0098	9
	10	0.5289	0.4099	0.2711	0.1222	0.1324	0.3328	0.5232	0.6440	0.0285	9
	11	0.4706	0.3589	0.2157	0.0836	0.0968	0.2722	0.4429	0.5923	0.0135	10
	12	0.6793	0.5547	0.3706	0.1569	0.1855	0.4459	0.6516	0.7963	0.0279	10
	13	0.8220	0.7153	0.5140	0.2426	0.2707	0.6089	0.8019	0.9080	0.0444	10
	14	0.7816	0.6584	0.4503	0.1930	0.2218	0.5409	0.7677	0.8818	0.0288	11
	15	0.8746	0.7794	0.5769	0.2701	0.3051	0.6746	0.8673	0.9491	0.0400	11

k	M	λ									D1
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1	
5	8	0.3666	0.3024	0.2064	0.1021	0.1338	0.3063	0.4608	0.5979	0.0496	8
	9	0.3020	0.2355	0.1484	0.0686	0.0919	0.2390	0.3791	0.5233	0.0242	9
	10	0.2615	0.1883	0.1102	0.0396	0.0578	0.1850	0.3200	0.4501	0.0118	10
	11	0.5458	0.4404	0.2865	0.1159	0.1702	0.4324	0.6417	0.7836	0.0368	10
	12	0.4798	0.3847	0.2252	0.0792	0.1292	0.3591	0.5736	0.7354	0.0196	11
	13	0.7044	0.5884	0.3989	0.1529	0.2333	0.5757	0.7873	0.9055	0.0396	11
	14	0.6477	0.5304	0.3361	0.1130	0.1736	0.4972	0.7364	0.8792	0.0216	12
	15	0.8025	0.7056	0.4923	0.1879	0.2789	0.6697	0.8729	0.9602	0.0352	12

k	M	λ									D1
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1	
7	8	-	-	-	-	-	-	-	-	-	-
	9	0.3842	0.2441	0.1638	0.1060	0.1154	0.3074	0.5023	0.6541	0.0435	9
	10	0.3277	0.1878	0.1078	0.0690	0.0756	0.2372	0.4240	0.5863	0.0199	10
	11	0.2708	0.1412	0.0820	0.0446	0.0517	0.1817	0.3600	0.5188	0.0094	11
	12	0.5823	0.3618	0.2213	0.1346	0.1484	0.4399	0.7030	0.8514	0.0333	11
	13	0.5234	0.3019	0.1653	0.0948	0.1009	0.3658	0.6399	0.8026	0.0159	12
	14	0.7553	0.5021	0.3158	0.1875	0.2002	0.5921	0.8473	0.9490	0.0320	12
	15	0.7116	0.4456	0.2494	0.1387	0.1466	0.5106	0.7990	0.9290	0.0169	13

(c) N = 1680

k	M	λ								D1
		0.199	0.333	0.5	2	3	4	5	1	
3	8	0.3056	0.1445	0.0695	0.0763	0.1812	0.2986	0.3984	0.0213	8
	9	0.2547	0.1103	0.0453	0.0531	0.1402	0.2424	0.3403	0.0112	9
	10	0.5264	0.2775	0.1272	0.1334	0.3374	0.5116	0.6550	0.0279	9
	11	0.4706	0.2206	0.0846	0.1011	0.2836	0.4524	0.6015	0.0183	10
	12	0.6790	0.3771	0.1674	0.1852	0.4604	0.6625	0.7955	0.0334	10
	13	0.6316	0.3222	0.1265	0.1470	0.3950	0.5991	0.7530	0.0181	11
	14	0.7803	0.4644	0.1968	0.2323	0.5376	0.7585	0.8790	0.0320	11
	15	0.8730	0.5800	0.2759	0.3149	0.6778	0.8584	0.9466	0.0506	11

k	M	λ								D1
		0.2	0.33	0.504	2	3	4	4.98	1	
5	8	-	-	-	-	-	-	-	-	-
	9	0.3196	0.1631	0.0774	0.0988	0.2501	0.4054	0.5381	0.0305	9
	10	0.2621	0.1189	0.0491	0.0627	0.1918	0.3356	0.4666	0.0154	10
	11	0.5503	0.3051	0.1342	0.1791	0.4373	0.6635	0.8071	0.0444	10
	12	0.4961	0.2434	0.0993	0.1333	0.3728	0.5905	0.7412	0.0249	11
	13	0.4458	0.2004	0.0703	0.0998	0.3176	0.5247	0.6980	0.0146	12
	14	0.6569	0.3523	0.1407	0.1877	0.5181	0.7513	0.8891	0.0301	12
	15	0.6141	0.2983	0.1059	0.1455	0.4550	0.7048	0.8541	0.0176	13

(d) N = 1680000

k	M	λ									D1
		0.2	0.25	0.333	0.5	2	3	4	5	1	
3	8	0.3046	0.2346	0.1496	0.0676	0.0791	0.1840	0.2919	0.3949	0.0234	8
	9	0.2543	0.1838	0.1150	0.0443	0.0509	0.1381	0.2385	0.3373	0.0101	9
	10	0.5283	0.4118	0.2739	0.1260	0.1380	0.3320	0.5114	0.6464	0.0314	9
	11	0.4703	0.3524	0.2246	0.0950	0.1057	0.2785	0.4499	0.5923	0.0173	10
	12	0.6785	0.5534	0.3824	0.1718	0.1909	0.4574	0.6625	0.7978	0.0335	10
	13	0.6239	0.4984	0.3242	0.1237	0.1467	0.3954	0.6124	0.7510	0.0200	11
	14	0.7773	0.6565	0.4644	0.1999	0.2315	0.5500	0.7638	0.8815	0.0348	11
	15	0.7426	0.6008	0.4064	0.1636	0.1807	0.4888	0.7216	0.8522	0.0216	12

k	M	λ									D1
		0.2	0.25	0.333	0.5	2	3	4	5	1	
5	8	-	-	-	-	-	-	-	-	-	-
	9	0.3220	0.2491	0.1604	0.0819	0.1010	0.2519	0.4032	0.5439	0.0298	9
	10	0.2697	0.2019	0.1244	0.0531	0.0719	0.1940	0.3389	0.4706	0.0166	10
	11	0.5531	0.4443	0.3123	0.1481	0.1923	0.4508	0.6659	0.8048	0.0495	10
	12	0.5002	0.3897	0.2486	0.1065	0.1326	0.3754	0.5988	0.7518	0.0269	11
	13	0.4516	0.3324	0.2039	0.0762	0.0994	0.3139	0.5344	0.7012	0.0134	12
	14	0.6605	0.5337	0.3591	0.1487	0.1959	0.5160	0.7610	0.8910	0.0334	12
	15	0.6161	0.4808	0.3001	0.1120	0.1512	0.4498	0.7090	0.8528	0.0203	13

Table 4.4: Expected sample size for Procedure 1

(a) N = 60

k	M	λ								
		0.222	0.222	0.308	0.5	2	3	4	4.67	1
3	8	14.54	14.54	15.10	16.33	15.42	13.17	11.90	11.35	17.94
	9	16.62	16.62	17.26	18.69	17.49	14.83	13.39	12.77	20.71
	10	18.72	18.72	19.44	21.07	19.54	16.48	14.88	14.19	23.53
	11	20.83	20.83	21.64	23.47	21.56	18.13	16.37	15.60	26.41
	12	22.97	22.97	23.87	25.89	23.57	19.78	17.85	17.02	29.35
	13	25.12	25.12	26.11	28.33	25.56	21.43	19.34	18.44	32.33
	14	27.29	27.29	28.37	30.80	27.54	23.08	20.83	19.86	35.38
	15	29.48	29.48	30.64	33.28	29.52	24.73	22.32	21.28	38.49

k	M	λ								
		0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1
5	8	24.61	24.61	24.61	26.67	22.85	19.48	16.83	14.79	28.02
	9	28.73	28.73	28.73	31.20	25.98	21.95	18.93	16.64	33.03
	10	33.01	33.01	33.01	35.92	29.00	24.40	21.03	18.48	38.36
	11	37.47	37.47	37.47	40.90	31.95	26.84	23.14	20.33	44.13
	12	42.16	42.16	42.16	46.22	34.86	29.28	25.24	22.18	50.68
	13	47.16	47.16	47.16	52.18	37.76	31.72	27.34	24.03	-
	14	52.75	52.75	52.75	-	40.67	34.16	29.45	25.88	-
	15	-	-	-	-	43.57	36.60	31.55	27.73	-

k	M	λ								
		0	0.455	0.455	0.455	1.67	2.5	3.57	5	1
6	8	28.02	30.96	30.96	30.96	29.48	23.20	18.77	15.74	33.68
	9	33.03	36.64	36.64	36.64	33.94	26.14	21.12	17.71	40.36
	10	38.36	42.82	42.82	42.82	38.13	29.05	23.46	19.68	48.08
	11	44.13	49.85	49.85	49.85	41.94	31.95	25.81	21.65	-
	12	50.68	-	-	-	45.75	34.86	28.15	23.61	-
	13	-	-	-	-	49.56	37.76	30.50	25.58	-
	14	-	-	-	-	53.38	40.67	32.85	27.55	-
	15	-	-	-	-	57.19	43.57	35.19	29.52	-

(b) N = 420

k	M	λ								
		0.199	0.246	0.333	0.5	2	3	4	5	1
3	8	14.17	14.47	15.02	15.99	15.26	13.21	11.97	11.18	17.29
	9	16.16	16.51	17.14	18.28	17.35	14.90	13.47	12.59	19.88
	10	18.17	18.55	19.27	20.57	19.42	16.59	14.98	13.99	22.49
	11	20.18	20.61	21.41	22.88	21.49	18.27	16.48	15.38	25.12
	12	22.21	22.68	23.56	25.20	23.55	19.94	17.98	16.78	27.77
	13	24.24	24.76	25.72	27.52	25.60	21.62	19.48	18.18	30.44
	14	26.28	26.84	27.89	29.85	27.64	23.28	20.97	19.58	33.13
	15	28.33	28.94	30.06	32.19	29.68	24.95	22.47	20.98	35.83

k	M	λ								
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1
5	8	22.52	22.75	23.22	24.17	22.18	18.35	16.06	14.44	25.42
	9	26.03	26.29	26.84	27.95	25.38	20.76	18.10	16.26	29.50
	10	29.58	29.88	30.50	31.79	28.56	23.14	20.13	18.07	33.64
	11	33.17	33.51	34.20	35.66	31.72	25.50	22.15	19.87	37.84
	12	36.80	37.17	37.95	39.58	34.87	27.85	24.17	21.68	42.09
	13	40.46	40.87	41.72	43.52	37.99	30.20	26.18	23.49	46.38
	14	44.15	44.60	45.53	47.50	41.10	32.54	28.20	25.30	50.72
	15	47.86	48.35	49.36	51.50	44.20	34.87	30.21	27.10	55.09

k	M	λ								
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1
7	8	30.17	30.62	31.09	31.56	29.36	23.74	19.85	17.43	33.08
	9	35.13	35.66	36.21	36.77	33.81	26.92	22.37	19.62	38.63
	10	40.18	40.79	41.42	42.06	38.25	30.05	24.89	21.81	44.29
	11	45.31	46.00	46.71	47.44	42.67	33.15	27.39	23.99	50.05
	12	50.51	51.28	52.07	52.89	47.06	36.24	29.89	26.18	55.90
	13	55.77	56.62	57.50	58.41	51.43	39.30	32.38	28.36	61.83
	14	61.09	62.03	62.99	63.99	55.77	42.36	34.87	30.54	67.83
	15	66.47	67.49	68.54	69.62	60.08	45.40	37.37	32.72	73.90

(c) N = 1680

k	M	λ								
		0.199	0.249	0.333	0.5	2	3	4	5	1
3	8	14.14	14.46	14.85	15.95	15.24	13.21	11.97	11.19	17.23
	9	16.13	16.50	16.92	18.23	17.32	14.91	13.48	12.59	19.79
	10	18.13	18.54	18.99	20.52	19.40	16.60	14.99	13.99	22.38
	11	20.14	20.59	21.07	22.81	21.47	18.28	16.49	15.40	24.99
	12	22.15	22.66	23.15	25.12	23.53	19.96	17.99	16.80	27.62
	13	24.18	24.73	25.24	27.43	25.58	21.63	19.49	18.20	30.26
	14	26.21	26.80	27.32	29.75	27.63	23.31	20.99	19.60	32.92
	15	28.25	28.89	29.41	32.08	29.67	24.98	22.49	20.99	35.60

k	M	λ								
		0.2	0.253	0.33	0.504	2	3	4	4.98	1
5	8	22.37	22.65	23.06	23.56	22.09	18.35	15.93	14.40	25.19
	9	25.83	26.16	26.63	27.17	25.28	20.76	17.95	16.21	29.20
	10	29.34	29.71	30.24	30.80	28.45	23.15	19.96	18.01	33.27
	11	32.88	33.30	33.90	34.46	31.61	25.53	21.97	19.82	37.39
	12	36.45	36.91	37.58	38.12	34.76	27.89	23.98	21.62	41.55
	13	40.05	40.56	41.29	41.80	37.90	30.25	25.98	23.42	45.75
	14	43.68	44.23	45.03	45.48	41.02	32.60	27.98	25.22	49.98
	15	47.33	47.93	48.80	49.16	44.12	34.94	29.98	27.03	54.24

(d) N = 1680000

k	M	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
3	8	14.14	14.46	14.98	15.94	15.23	13.21	11.97	11.19	17.20
	9	16.13	16.49	17.09	18.21	17.32	14.91	13.48	12.60	19.76
	10	18.12	18.53	19.21	20.50	19.39	16.60	14.99	14.00	22.34
	11	20.13	20.59	21.34	22.79	21.46	18.28	16.49	15.40	24.95
	12	22.14	22.65	23.48	25.10	23.52	19.96	18.00	16.80	27.57
	13	24.17	24.72	25.63	27.40	25.57	21.64	19.50	18.20	30.21
	14	26.20	26.79	27.78	29.72	27.62	23.31	21.00	19.60	32.86
	15	28.23	28.87	29.94	32.04	29.67	24.99	22.50	21.00	35.52

k	M	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
5	8	22.32	22.58	23.02	23.85	22.06	18.35	15.93	14.38	25.11
	9	25.77	26.08	26.58	27.56	25.24	20.76	17.96	16.19	29.10
	10	29.26	29.61	30.19	31.31	28.42	23.16	19.97	17.99	33.15
	11	32.78	33.18	33.82	35.09	31.58	25.53	21.98	19.80	37.24
	12	36.34	36.77	37.49	38.91	34.73	27.90	23.99	21.60	41.38
	13	39.92	40.40	41.19	42.75	37.86	30.26	25.99	23.40	45.55
	14	43.53	44.05	44.91	46.62	40.98	32.61	28.00	25.20	49.75
	15	47.16	47.72	48.66	50.51	44.09	34.96	30.00	27.00	53.98

Table 4.5: Simulated sample size for Procedure 1

(a) N = 60

k	M	λ								
		0.222	0.222	0.308	0.5	2	3	4	4.67	1
3	8	14.55	14.54	15.10	16.33	15.44	13.16	11.92	11.34	17.94
	9	16.60	16.61	17.26	18.71	17.45	14.85	13.38	12.77	20.68
	10	18.72	18.72	19.45	21.05	19.50	16.48	14.89	14.20	23.53
	11	20.83	20.82	21.66	23.49	21.55	18.13	16.38	15.60	26.42
	12	22.96	22.96	23.85	25.89	23.56	19.74	17.85	17.02	29.37
	13	25.12	25.11	26.12	28.31	25.54	21.40	19.34	18.43	32.29
	14	27.28	27.27	28.38	30.78	27.55	23.09	20.80	19.85	35.39
	15	29.50	29.48	30.64	33.25	29.55	24.76	22.30	21.25	38.48

k	M	λ								
		0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1
5	8	24.65	24.64	24.61	26.68	22.84	19.46	16.83	14.76	28.00
	9	28.80	28.73	28.69	31.20	25.95	21.98	18.98	16.65	33.01
	10	33.03	33.00	33.03	35.93	29.06	24.37	21.03	18.48	38.34
	11	37.49	37.46	37.48	40.89	31.95	26.91	23.13	20.36	44.10
	12	42.16	42.18	42.16	46.15	34.86	29.25	25.24	22.17	50.67
	13	47.15	47.18	47.13	52.18	37.71	31.74	27.33	24.03	-
	14	52.74	52.71	52.77	-	40.68	34.17	29.48	25.89	-
	15	-	-	-	-	43.50	36.66	31.55	27.69	-

k	M	λ								
		0	0.455	0.455	0.455	1.67	2.5	3.57	5	1
6	8	28.01	30.96	30.95	30.91	29.49	23.18	18.75	15.73	33.69
	9	33.06	36.69	36.62	36.62	33.89	26.16	21.14	17.72	40.41
	10	38.37	42.83	42.80	42.79	38.11	29.05	23.41	19.73	48.11
	11	44.13	49.90	49.86	49.83	41.93	31.97	25.80	21.61	-
	12	50.67	-	-	-	45.76	34.78	28.19	23.64	-
	13	-	-	-	-	49.54	37.75	30.48	25.56	-
	14	-	-	-	-	53.40	40.69	32.88	27.60	-
	15	-	-	-	-	57.22	43.63	35.19	29.56	-

(b) N = 420

k	M	λ								
		0.199	0.246	0.333	0.5	2	3	4	5	1
3	8	14.16	14.47	15.01	15.99	15.28	13.20	12.00	11.17	17.26
	9	16.15	16.49	17.17	18.27	17.33	14.88	13.50	12.59	19.86
	10	18.14	18.56	19.27	20.59	19.41	16.59	14.96	13.98	22.48
	11	20.20	20.59	21.45	22.92	21.44	18.26	16.45	15.38	25.13
	12	22.20	22.71	23.53	25.21	23.53	19.99	17.97	16.79	27.76
	13	24.26	24.78	25.71	27.46	25.65	21.63	19.46	18.17	30.42
	14	26.28	26.81	27.85	29.78	27.64	23.28	20.97	19.59	33.21
	15	28.32	28.90	30.10	32.17	29.68	24.92	22.45	20.96	35.80

k	M	λ								
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1
5	8	22.60	22.77	23.23	24.16	22.26	18.40	16.03	14.44	25.40
	9	26.01	26.28	26.82	27.95	25.32	20.75	18.07	16.28	29.54
	10	29.62	29.92	30.54	31.81	28.48	23.13	20.12	18.07	33.64
	11	33.14	33.52	34.22	35.65	31.68	25.50	22.10	19.95	37.82
	12	36.84	37.18	37.94	39.60	34.74	27.90	24.13	21.71	42.11
	13	40.46	40.93	41.72	43.43	37.97	30.18	26.21	23.46	46.41
	14	44.14	44.56	45.49	47.56	41.11	32.56	28.16	25.28	50.68
	15	47.87	48.27	49.30	51.49	44.20	34.88	30.23	27.11	55.14

k	M	λ								
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1
7	8	30.19	30.60	31.12	31.56	29.25	23.69	19.86	17.45	33.07
	9	35.01	35.61	36.16	36.80	33.81	26.96	22.32	19.59	38.63
	10	40.22	40.78	41.33	42.09	38.20	29.97	24.85	21.79	44.27
	11	45.25	46.02	46.71	47.46	42.65	33.16	27.34	24.00	50.13
	12	50.53	51.30	52.05	52.88	47.10	36.28	29.93	26.15	55.81
	13	55.82	56.59	57.40	58.36	51.41	39.20	32.37	28.30	61.84
	14	61.13	62.09	62.94	64.07	55.82	42.36	34.84	30.55	67.83
	15	66.51	67.46	68.51	69.65	60.03	45.40	37.41	32.67	73.88

(c) N = 1680

k	M	λ								
		0.199	0.249	0.333	0.5	2	3	4	5	1
3	8	14.16	14.47	14.98	15.94	15.27	13.20	11.95	11.19	17.23
	9	16.14	16.51	17.10	18.25	17.32	14.89	13.45	12.60	19.78
	10	18.13	18.56	19.22	20.52	19.38	16.58	14.98	14.01	22.37
	11	20.14	20.61	21.40	22.78	21.47	18.30	16.51	15.42	25.01
	12	22.16	22.65	23.48	25.14	23.53	19.94	17.98	16.80	27.64
	13	24.21	24.73	25.64	27.45	25.58	21.63	19.53	18.19	30.25
	14	26.25	26.85	27.83	29.78	27.61	23.30	21.00	19.61	32.97
	15	28.27	28.86	30.01	32.13	29.67	25.04	22.49	21.00	35.58

k	M	λ								
		0.2	0.253	0.33	0.504	2	3	4	4.98	1
5	8	22.38	22.62	23.05	23.92	22.06	18.35	15.92	14.37	25.21
	9	25.83	26.16	26.62	27.60	25.32	20.75	17.99	16.20	29.17
	10	29.34	29.69	30.25	31.42	28.34	23.14	20.00	18.02	33.19
	11	32.85	33.28	33.90	35.22	31.60	25.55	21.97	19.82	37.35
	12	36.44	36.97	37.58	39.12	34.70	27.90	23.95	21.62	41.58
	13	40.08	40.56	41.34	42.91	37.89	30.20	26.00	23.48	45.74
	14	43.70	44.22	44.94	46.84	41.08	32.61	28.01	25.23	49.98
	15	47.34	47.89	48.77	50.72	44.08	34.93	29.99	26.96	54.28

(d) N = 1680000

k	M	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
3	8	14.14	14.44	15.02	15.91	15.20	13.19	11.99	11.17	17.21
	9	16.13	16.50	17.06	18.22	17.30	14.92	13.48	12.60	19.76
	10	18.12	18.51	19.19	20.48	19.40	16.59	14.99	13.96	22.36
	11	20.12	20.57	21.35	22.81	21.43	18.24	16.50	15.40	24.96
	12	22.17	22.68	23.50	25.08	23.46	20.01	18.04	16.81	27.56
	13	24.16	24.75	25.57	27.41	25.60	21.65	19.43	18.19	30.24
	14	26.24	26.84	27.77	29.66	27.65	23.30	20.98	19.58	32.82
	15	28.21	28.85	29.93	32.06	29.66	24.96	22.51	20.99	35.53

k	M	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
5	8	22.30	22.54	23.02	23.82	22.00	18.35	15.94	14.36	25.08
	9	25.77	26.08	26.60	27.52	25.20	20.80	17.95	16.23	29.19
	10	29.26	29.63	30.19	31.34	28.42	23.18	19.97	17.99	33.15
	11	32.77	33.15	33.85	35.11	31.57	25.60	21.96	19.78	37.19
	12	36.38	36.75	37.47	38.93	34.71	27.93	24.03	21.67	41.35
	13	39.96	40.44	41.17	42.67	37.85	30.35	25.98	23.47	45.62
	14	43.57	44.09	44.91	46.69	40.94	32.53	28.04	25.17	49.77
	15	47.16	47.76	48.73	50.56	44.11	34.93	29.93	26.98	53.90

4.2 Powers of Procedure 2

The powers of Procedure 2 provide a method of checking the effectiveness of Procedure 2 for testing the null hypothesis. Powers of Procedure 2 are the probabilities of $W_{kt} \geq D_2$ under the alternative hypothesis H_a. Similar to the results in chapter 3, we need to calculate the target probabilities in different layers. To get the exact powers of Procedure 2, we need to calculate transfer probabilities under H_a. Since at the current time a single personal computer is too slow to calculate the transfer probabilities under H_a, we didn't do the exact calculation of powers of Procedure 2 and thus don't give the formula for these transfer probabilities under H_a. In the future, we may produce these formulae and do the exact calculations under H_a based on the methods of chapter 3 and the following theorem.

Theorem 4.1: Under the alternative hypothesis H_a, the target probability in layer N_1 is given by

$$
\begin{aligned}
C_{N_1} = \; & \sum_{\substack{x_{k-1}=0 \\ x_k=x_{k-1}+D_2-1 \\ x_{k-1}=x_{k-2}=\ldots=x_{k-j} \\ x_1+x_2+\ldots+x_k=N_1}}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} \cdots \sum_{x_1=0}^{x_{k-1}-1} B_k(x_1, \ldots, x_{k-1}, x_k) \\
& +(k-1)\{\sum_{\substack{x_2=0 \\ x_1=x_2+D_2-1 \\ x_2=x_3=\ldots=x_{1+j} \\ x_1+x_2+\ldots+x_k=N_1}}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=1}^{k-2} \binom{k-2}{j} \sum_{x_{2+j}=0}^{x_2-1} \sum_{x_{3+j}=0}^{x_2-1} \cdots \sum_{x_k=0}^{x_2-1} B_1(x_1, \ldots, x_{k-1}, x_k) \\
& + \sum_{\substack{x_k=0 \\ x_1=x_k+D_2-1 \\ x_k=x_{k-1}=\ldots=x_{k-j} \\ x_1+x_2+\ldots+x_k=N_1}}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=0}^{k-2} \binom{k-2}{j} \sum_{x_{k-j-1}=0}^{x_k-1} \sum_{x_{k-j-2}=0}^{x_k-1} \cdots \sum_{x_2=0}^{x_k-1} B_1(x_1, \ldots, x_{k-1}, x_k)\},
\end{aligned}
$$

Proof: See Appendix A.2.1

4.3 Approximation for $P_{H_a}(W_{kt} \geq D_2)$ and Sample Size

Since calculations of transfer probabilities under H_a would require a heavy computational effort and time, in order to speed up the calculations we need to sacrifice the precisions for quickly getting the results. The process is the same as the approximation of P_{H_0}, hence here we just give the formula.

$$P_{H_a}(X_{(k)t} - X_{(k-1)t} \geq D_2) = \sum_{i=D_2-1}^{N_0-1} C_i - \sum_{D_2-1 \leq i_1 < i_2 < N_0} C_{i_1} C_{i_2} + ... + (-1)^{r+1} \sum_{D_2-1 \leq i_1 < i_2 < ... < i_r < N_0} C_{i_1} C_{i_2} ... C_{i_r}, \qquad (4.1)$$

where the C_i's are the target probabilities in layer i under H_a, for $i = D_2 - 1, D_2, ..., N_0 - 1$.

For procedure 2, the approximation of sample size $E(\omega)$ can be obtained by the following procedures:

From chapter 3.4 we know that

$$\begin{aligned} P_{N_1} = Pr((\tilde{\cup}_{i=D_2-1}^{N_1} S_i)) &\approx Pr(\tilde{\cup}_{i=D_2-1}^{N_1-1} S_i) + (1 - Pr(\tilde{\cup}_{i=D_2-1}^{N_1-1} S_i)) C_{N_1} \\ &= P_{N_1-1} + (1 - P_{N_1-1}) C_{N_1}, \end{aligned}$$

where S_i is the target event set at layer i for $i = D_2 - 1, D_2, ..., N_1$.

Therefore the probability of Procedure 2 stopping at population size $N_1 + 1$ removing the effects of the other layers is given as follows:

$$Pr(\omega = N_1 + 1) = P_{N_1} - P_{N_1-1}.$$

After $N_1 > N_0 - 1$, the probability of Procedure 2 stopping at population size N_0 without arriving the target point is

$$Pr'(\omega = N_0) = 1 - P_{N_0-1}.$$

Therefore,

$$E(\omega) = \sum_{N_1=D_2-1}^{N_0-1} Pr(\omega = N_1 + 1)(N_1 + 1) + (1 - P_{N_0-1})N_0$$

4.4 Results of Powers and Sample Size for Procedure 2

In tables 4.6 and 4.7, we present the simulated powers of Procedure 2 for values of λ as close as possible to $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 2, 3, 4, 5, 1$, $k = 3(2)7$, and $\alpha = 0.05$ and 0.1, respectively.

When $\lambda > 1$, the bigger λ is, the bigger the powers of Procedure 2 are and for $N = 60$ the powers of Procedure 2 increase more quickly as N_0 increases compared with those for the other N's for all values of k and λ. For example, looking at the column of $\lambda = 3$ within the group of $k = 5$ under $N = 60$, when $N_0 = 25$ the power is 0.5814 and when $N_0 = 40$ the power is 0.9705, while under $N = 1680$, when $N_0 = 25$ the power is 0.4949 and when $N_0 = 40$ the power is 0.6693. When $\lambda < 1$, all the powers of Procedure 2 are very small and even close to the powers under the null hypothesis. When $N \geq 1680000$ the results of Tables 4.6 and 4.7 are the same as Tables 8 and 9 in the paper of Panchapakesan et al. (1998).

Comparing Table 4.1 with Table 4.6 we found that when $\lambda > 1$ the powers of Procedure 1 are smaller than those of Procedure 2, i.e., Procedure 2 is clearly superior to Procedure 1 in power when $\lambda > 1$. But when $\lambda < 1$, the powers of Procedure 1 in Table 4.1 are seen to be distinctly bigger than the powers under the null hypothesis; however the powers of Procedure 2 are smaller than those of Procedure 1 and even close to the powers under the null hypothesis.

Approximate powers of Procedure 2 by Theorem 4.1 and approximate formula (4.1) at the 5% significance level are given in Table 4.8. Comparing the data in Table 4.6 with those in Table 4.8 we can see that the two methods get very close results for all values of k, λ, N_0 and D_2.

The simulated sample sizes of Procedure 2 at the 5% and 10% significance levels are shown

in Tables 4.9 and 4.10, respectively. From Table 4.9, we can see that when $\lambda < 1$, the expected sample sizes of Procedure 2 are close among all N's for all values of λ, N_0 and D_2. When $\lambda > 1$, the smaller N is, the smaller the expected sample sizes of Procedure 2 are. Comparing Table 4.9 with Table 4.4, we see that when $\lambda > 1$ the expected sample sizes of Procedure 2 are smaller than those of Procedure 1 for all values of k under the same powers of the procedures with $\alpha = 0.05$. When $N \geq 1680000$ the results of Tables 4.9 and 4.10 are the same as Tables 10 and 11 in the paper of Panchapakesan et al. (1998).

Table 4.6: Simulated powers of Procedure 2 at the 5% significance level

(a) N = 60

k	N_0	λ									D2
		0.222	0.222	0.308	0.5	2	3	4	4.67	1	
3	18	0.0913	0.0910	0.0762	0.0473	0.1619	0.4770	0.7234	0.8260	0.0139	7
	21	0.1237	0.1241	0.1078	0.0706	0.2527	0.6447	0.8680	0.9365	0.0213	7
	24	0.1557	0.1561	0.1396	0.0955	0.3488	0.7828	0.9488	0.9820	0.0286	7
	27	0.1867	0.1858	0.1685	0.1219	0.4535	0.8816	0.9842	0.9967	0.0355	7
	30	0.1183	0.1184	0.1056	0.0726	0.4277	0.8942	0.9906	0.9985	0.0145	8
	33	0.1334	0.1328	0.1203	0.0850	0.5266	0.9534	0.9985	1.0000	0.0167	8
	36	0.1492	0.1497	0.1336	0.0962	0.6320	0.9844	0.9999	1.0000	0.0185	8

k	N_0	λ									D2
		0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1	
5	25	0.0593	0.0602	0.0589	0.0419	0.2700	0.5814	0.8436	0.9671	0.0277	5
	30	0.0754	0.0756	0.0757	0.0546	0.4017	0.7604	0.9535	0.9968	0.0343	5
	35	0.0886	0.0882	0.0891	0.0643	0.5492	0.8946	0.9918	0.9999	0.0387	5
	40	0.0978	0.0963	0.0973	0.0699	0.7025	0.9705	0.9996	1.0000	0.0409	5
	45	0.0992	0.1002	0.1005	0.0729	0.8442	0.9969	1.0000	1.0000	0.0408	5
	50	0.0994	0.0998	0.1005	0.0718	0.9540	1.0000	1.0000	1.0000	0.0408	5
	55	0.0253	0.0253	0.0253	0.0154	0.9904	1.0000	1.0000	1.0000	0.0073	6

k	N_0	λ									D2
		0	0.455	0.455	0.455	1.67	2.5	3.57	5	1	
6	29	0.0326	0.0224	0.0224	0.0227	0.1003	0.4957	0.8789	0.9915	0.0120	5
	35	0.0392	0.0260	0.0265	0.0263	0.1621	0.7056	0.9770	0.9999	0.0128	5
	41	0.0406	0.0272	0.0268	0.0285	0.2475	0.8811	0.9989	1.0000	0.0130	5
	47	0.0406	0.0276	0.0274	0.0271	0.3681	0.9797	1.0000	1.0000	0.0126	5
	53	0.0414	0.0283	0.0271	0.0273	0.5585	1.0000	1.0000	1.0000	0.0129	5
	59	0.0403	0.0279	0.0277	0.0274	0.9503	1.0000	1.0000	1.0000	0.0132	5
	65	-	-	-	-	-	-	-	-	-	-

(b) N = 420

k	N_0	λ									D2
		0.199	0.246	0.333	0.5	2	3	4	5	1	
3	18	0.1386	0.1297	0.1111	0.0792	0.1891	0.4732	0.6918	0.8208	0.0348	7
	21	0.1105	0.1028	0.0887	0.0640	0.1867	0.5012	0.7290	0.8586	0.0238	8
	24	0.1503	0.1411	0.1262	0.0921	0.2650	0.6260	0.8362	0.9306	0.0365	8
	27	0.1183	0.1103	0.0965	0.0697	0.2463	0.6307	0.8518	0.9435	0.0232	9
	30	0.1474	0.1399	0.1239	0.0909	0.3201	0.7288	0.9122	0.9731	0.0320	9
	33	0.1754	0.1666	0.1529	0.1158	0.3913	0.8039	0.9503	0.9881	0.0415	9
	36	0.1369	0.1316	0.1177	0.0868	0.3671	0.8011	0.9543	0.9899	0.0271	10

k	N_0	λ									D2
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1	
5	25	0.0614	0.0594	0.0568	0.0480	0.1736	0.4995	0.7385	0.8816	0.0324	6
	30	0.0916	0.0884	0.0837	0.0734	0.2618	0.6485	0.8604	0.9539	0.0509	6
	35	0.0568	0.0563	0.0532	0.0456	0.2440	0.6668	0.8849	0.9682	0.0291	7
	40	0.0786	0.0762	0.0726	0.0634	0.3239	0.7738	0.9418	0.9890	0.0401	7
	45	0.0502	0.0480	0.0461	0.0391	0.3032	0.7834	0.9512	0.9923	0.0230	8
	50	0.0638	0.0612	0.0582	0.0508	0.3788	0.8561	0.9767	0.9973	0.0306	8
	55	0.0778	0.0756	0.0725	0.0624	0.4535	0.9073	0.9891	0.9992	0.0384	8

k	N_0	λ									D2
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1	
7	33	0.0312	0.0305	0.0285	0.0275	0.1144	0.4438	0.7523	0.9061	0.0194	6
	40	0.0503	0.0481	0.0461	0.0442	0.1844	0.6053	0.8796	0.9701	0.0320	6
	47	0.0689	0.0677	0.0655	0.0631	0.2627	0.7325	0.9452	0.9913	0.0453	6
	54	0.0389	0.0374	0.0359	0.0340	0.2376	0.7486	0.9590	0.9951	0.0234	7
	61	0.0522	0.0500	0.0475	0.0450	0.3105	0.8345	0.9828	0.9986	0.0322	7
	68	0.0641	0.0619	0.0602	0.0574	0.3837	0.8970	0.9933	0.9997	0.0404	7
	75	0.0357	0.0347	0.0334	0.0315	0.3580	0.9036	0.9952	0.9998	0.0210	8

(c) N = 1680

k	N_0	λ									D2
		0.199	0.249	0.333	0.5	2	3	4	5	1	
3	18	0.1457	0.1338	0.1165	0.0836	0.1938	0.4720	0.6874	0.8191	0.0382	7
	21	0.1177	0.1099	0.0947	0.0672	0.1910	0.4999	0.7261	0.8524	0.0275	8
	24	0.1578	0.1479	0.1316	0.0988	0.2663	0.6229	0.8313	0.9260	0.0404	8
	27	0.1258	0.1171	0.1040	0.0739	0.2507	0.6242	0.8432	0.9370	0.0259	9
	30	0.1558	0.1480	0.1331	0.0995	0.3222	0.7223	0.9059	0.9698	0.0368	9
	33	0.1238	0.1163	0.1052	0.0764	0.2991	0.7202	0.9125	0.9731	0.0249	10
	36	0.1484	0.1414	0.1285	0.0958	0.3663	0.7917	0.9473	0.9869	0.0320	10

k	N_0	λ									D2
		0.2	0.253	0.33	0.504	2	3	4	4.98	1	
5	25	0.0672	0.0651	0.0624	0.0538	0.1787	0.4949	0.7459	0.8802	0.0366	6
	30	0.0462	0.0437	0.0420	0.0352	0.1708	0.5241	0.7872	0.9142	0.0229	7
	35	0.0676	0.0640	0.0615	0.0537	0.2441	0.6552	0.8832	0.9646	0.0344	7
	40	0.0441	0.0437	0.0406	0.0354	0.2323	0.6693	0.9024	0.9741	0.0209	8
	45	0.0591	0.0580	0.0549	0.0485	0.3032	0.7649	0.9475	0.9892	0.0307	8
	50	0.0766	0.0744	0.0708	0.0627	0.3757	0.8371	0.9732	0.9960	0.0402	8
	55	0.0951	0.0913	0.0880	0.0785	0.4464	0.8899	0.9862	0.9985	0.0505	8

k	N_0	λ									D2
		0.199	0.245	0.34	0.512	2	2.98	4	4.98	1	
7	33	0.0373	0.0376	0.0356	0.0318	0.1385	0.4541	0.7382	0.8900	0.0248	6
	40	0.0592	0.0571	0.0555	0.0503	0.2142	0.6092	0.8643	0.9576	0.0396	6
	47	0.0359	0.0359	0.0344	0.0313	0.2033	0.6306	0.8923	0.9731	0.0223	7
	54	0.0504	0.0494	0.0485	0.0443	0.2779	0.7430	0.9468	0.9907	0.0337	7
	61	0.0674	0.0661	0.0647	0.0589	0.3505	0.8255	0.9751	0.9966	0.0447	7
	68	0.0417	0.0413	0.0394	0.0357	0.3295	0.8309	0.9796	0.9980	0.0259	8
	75	0.0518	0.0518	0.0506	0.0461	0.3988	0.8873	0.9907	0.9993	0.0335	8

(d) N = 1680000

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
3	18	0.1449	0.1357	0.1177	0.0846	0.1940	0.4720	0.6849	0.8175	0.0382	7
	21	0.1188	0.1118	0.0976	0.0688	0.1931	0.4974	0.7227	0.8518	0.0279	8
	24	0.1600	0.1494	0.1344	0.0997	0.2691	0.6219	0.8273	0.9247	0.0421	8
	27	0.1273	0.1203	0.1062	0.0762	0.2542	0.6228	0.8418	0.9351	0.0279	9
	30	0.1583	0.1506	0.1361	0.1008	0.3216	0.7198	0.9044	0.9681	0.0386	9
	33	0.1268	0.1207	0.1073	0.0792	0.3023	0.7162	0.9098	0.9716	0.0261	10
	36	0.1522	0.1460	0.1316	0.0991	0.3668	0.7883	0.9446	0.9861	0.0348	10

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
5	25	0.0697	0.0678	0.0637	0.0553	0.1814	0.4957	0.7423	0.8788	0.0385	6
	30	0.0471	0.0462	0.0434	0.0384	0.1707	0.5208	0.7837	0.9118	0.0242	7
	35	0.0698	0.0681	0.0637	0.0564	0.2468	0.6512	0.8791	0.9632	0.0374	7
	40	0.0470	0.0451	0.0423	0.0383	0.2297	0.6665	0.8978	0.9727	0.0227	8
	45	0.0636	0.0613	0.0583	0.0517	0.3043	0.7597	0.9448	0.9887	0.0322	8
	50	0.0428	0.0421	0.0400	0.0345	0.2883	0.7678	0.9524	0.9916	0.0200	9
	55	0.0563	0.0546	0.0513	0.0441	0.3527	0.8354	0.9750	0.9967	0.0267	9

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
7	33	0.0393	0.0384	0.0379	0.0348	0.1395	0.4586	0.7369	0.8864	0.0268	6
	40	0.0623	0.0618	0.0598	0.0544	0.2160	0.6084	0.8617	0.9570	0.0430	6
	47	0.0386	0.0387	0.0369	0.0338	0.2040	0.6322	0.8868	0.9713	0.0250	7
	54	0.0546	0.0538	0.0526	0.0489	0.2751	0.7399	0.9407	0.9896	0.0370	7
	61	0.0348	0.0342	0.0332	0.0298	0.2583	0.7535	0.9525	0.9930	0.0212	8
	68	0.0466	0.0441	0.0443	0.0400	0.3271	0.8283	0.9758	0.9974	0.0289	8
	75	0.0579	0.0582	0.0557	0.0518	0.3933	0.8830	0.9885	0.9991	0.0379	8

Table 4.7: Simulated powers of Procedure 2 at the 10% significance level

(a) N = 60

k	N_0	λ									D2
		0.222	0.222	0.308	0.5	2	3	4	4.67	1	
3	18	0.1829	0.1846	0.1616	0.1134	0.2851	0.6369	0.8447	0.9143	0.0463	6
	21	0.2323	0.2335	0.2111	0.1534	0.3910	0.7793	0.9361	0.9733	0.0629	6
	24	0.2778	0.2791	0.2559	0.1959	0.4983	0.8776	0.9784	0.9938	0.0788	6
	27	0.1863	0.1865	0.1683	0.1202	0.4519	0.8820	0.9846	0.9967	0.0352	7
	30	0.2143	0.2146	0.1962	0.1453	0.5574	0.9448	0.9967	0.9996	0.0422	7
	33	0.2369	0.2393	0.2178	0.1669	0.6575	0.9788	0.9996	1.0000	0.0475	7
	36	0.2602	0.2590	0.2401	0.1865	0.7482	0.9939	1.0000	1.0000	0.0516	7

k	N_0	λ									D2
		0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1	
5	25	0.0604	0.0600	0.0605	0.0417	0.2687	0.5814	0.8441	0.9668	0.0274	5
	30	0.0752	0.0764	0.0764	0.0541	0.4014	0.7624	0.9536	0.9964	0.0343	5
	35	0.0890	0.0884	0.0887	0.0636	0.5494	0.8938	0.9922	0.9999	0.0393	5
	40	0.0969	0.0961	0.0955	0.0702	0.6999	0.9705	0.9996	1.0000	0.0402	5
	45	0.1012	0.1012	0.1003	0.0720	0.8428	0.9968	1.0000	1.0000	0.0414	5
	50	0.1006	0.1000	0.0999	0.0724	0.9546	1.0000	1.0000	1.0000	0.0407	5
	55	0.0994	0.0999	0.0995	0.0725	0.9993	1.0000	1.0000	1.0000	0.0403	5

k	N_0	λ									D2
		0	0.455	0.455	0.455	1.67	2.5	3.57	5	1	
6	29	0.1361	0.1057	0.1075	0.1070	0.2338	0.6764	0.9477	0.9978	0.0704	4
	35	0.1603	0.1265	0.1251	0.1244	0.3353	0.8457	0.9931	1.0000	0.0806	4
	41	0.1736	0.1364	0.1361	0.1369	0.4573	0.9545	0.9998	1.0000	0.0840	4
	47	0.1764	0.1381	0.1373	0.1376	0.6107	0.9953	1.0000	1.0000	0.0838	4
	53	0.1778	0.1369	0.1369	0.1372	0.8072	1.0000	1.0000	1.0000	0.0850	4
	59	0.1772	0.1369	0.1374	0.1374	1.0000	1.0000	1.0000	1.0000	0.0854	4
	65	-	-	-	-	-	-	-	-	-	-

(b) N = 420

k	N_0	λ									D2
		0.199	0.246	0.333	0.5	2	3	4	5	1	
3	18	0.2434	0.2332	0.2073	0.1597	0.3061	0.6141	0.8016	0.9002	0.0849	6
	21	0.1908	0.1782	0.1586	0.1184	0.2784	0.6123	0.8170	0.9156	0.0535	7
	24	0.2412	0.2302	0.2078	0.1597	0.3667	0.7240	0.8959	0.9615	0.0745	7
	27	0.1868	0.1812	0.1611	0.1228	0.3442	0.7273	0.9063	0.9686	0.0508	8
	30	0.2259	0.2165	0.1968	0.1543	0.4206	0.8077	0.9476	0.9862	0.0649	8
	33	0.2623	0.2505	0.2319	0.1865	0.4949	0.8669	0.9708	0.9940	0.0816	8
	36	0.2959	0.2866	0.2658	0.2183	0.5598	0.9082	0.9848	0.9972	0.0994	8

k	N_0	λ									D2
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1	
5	25	0.1454	0.1407	0.1336	0.1176	0.2889	0.6343	0.8349	0.9359	0.0891	5
	30	0.0917	0.0893	0.0845	0.0732	0.2613	0.6493	0.8618	0.9540	0.0507	6
	35	0.1224	0.1216	0.1150	0.1010	0.3522	0.7657	0.9310	0.9837	0.0710	6
	40	0.1566	0.1547	0.1456	0.1326	0.4399	0.8481	0.9668	0.9946	0.0937	6
	45	0.0999	0.0978	0.0925	0.0815	0.4059	0.8507	0.9716	0.9961	0.0532	7
	50	0.1223	0.1206	0.1128	0.1016	0.4845	0.9038	0.9875	0.9987	0.0678	7
	55	0.1442	0.1423	0.1351	0.1205	0.5568	0.9406	0.9943	0.9996	0.0815	7

k	N_0	λ									D2
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1	
7	33	0.0881	0.0849	0.0815	0.0776	0.2079	0.5807	0.8422	0.9490	0.0614	5
	40	0.1247	0.1202	0.1167	0.1114	0.2988	0.7188	0.9297	0.9850	0.0889	5
	47	0.0703	0.0679	0.0647	0.0617	0.2609	0.7332	0.9454	0.9911	0.0466	6
	54	0.0924	0.0889	0.0857	0.0815	0.3402	0.8274	0.9775	0.9977	0.0604	6
	61	0.1151	0.1098	0.1069	0.1034	0.4207	0.8920	0.9908	0.9994	0.0770	6
	68	0.0642	0.0622	0.0598	0.0570	0.3824	0.8988	0.9935	0.9997	0.0410	7
	75	0.0784	0.0754	0.0742	0.0707	0.4591	0.9377	0.9974	0.9999	0.0507	7

(c) N = 1680

k	N_0	λ									D2
		0.199	0.249	0.333	0.5	2	3	4	5	1	
3	18	0.1445	0.1344	0.1172	0.0828	0.1930	0.4719	0.6882	0.8193	0.0376	7
	21	0.1978	0.1859	0.1661	0.1255	0.2798	0.6085	0.8112	0.9105	0.0585	7
	24	0.2484	0.2382	0.2162	0.1668	0.3670	0.7193	0.8894	0.9566	0.0814	7
	27	0.1980	0.1869	0.1691	0.1296	0.3459	0.7236	0.8987	0.9644	0.0556	8
	30	0.2369	0.2248	0.2063	0.1623	0.4221	0.8005	0.9414	0.9828	0.0731	8
	33	0.1893	0.1791	0.1624	0.1251	0.3921	0.7964	0.9450	0.9859	0.0491	9
	36	0.2167	0.2077	0.1916	0.1523	0.4634	0.8509	0.9678	0.9932	0.0608	9

k	N_0	λ									D2
		0.2	0.253	0.33	0.504	2	3	4	4.98	1	
5	25	0.0681	0.0658	0.0623	0.0545	0.1787	0.4941	0.7450	0.8797	0.0372	6
	30	0.1002	0.0974	0.0933	0.0823	0.2633	0.6390	0.8609	0.9505	0.0585	6
	35	0.1357	0.1331	0.1279	0.1134	0.3531	0.7545	0.9287	0.9812	0.0812	6
	40	0.0897	0.0872	0.0834	0.0734	0.3240	0.7589	0.9397	0.9856	0.0488	7
	45	0.1147	0.1129	0.1064	0.0947	0.4025	0.8338	0.9689	0.9948	0.0646	7
	50	0.1401	0.1368	0.1321	0.1189	0.4758	0.8881	0.9844	0.9981	0.0816	7
	55	0.0946	0.0922	0.0884	0.0784	0.4471	0.8900	0.9861	0.9985	0.0507	8

k	N_0	λ									D2
		0.199	0.245	0.34	0.512	2	2.98	4	4.98	1	
7	33	0.0985	0.0971	0.0929	0.0866	0.2370	0.5851	0.8316	0.9375	0.0705	5
	40	0.0588	0.0587	0.0570	0.0510	0.2141	0.6086	0.8643	0.9591	0.0396	6
	47	0.0836	0.0826	0.0794	0.0744	0.2990	0.7303	0.9329	0.9856	0.0580	6
	54	0.1107	0.1082	0.1051	0.0977	0.3806	0.8207	0.9683	0.9952	0.0778	6
	61	0.0670	0.0664	0.0634	0.0593	0.3495	0.8247	0.9744	0.9967	0.0445	7
	68	0.0846	0.0830	0.0816	0.0740	0.4248	0.8844	0.9879	0.9989	0.0577	7
	75	0.1027	0.1019	0.0993	0.0918	0.4969	0.9246	0.9948	0.9996	0.0704	7

(d) N = 1680000

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
3	18	0.1460	0.1349	0.1178	0.0847	0.1946	0.4716	0.6859	0.8167	0.0387	7
	21	0.1976	0.1881	0.1687	0.1271	0.2820	0.6104	0.8097	0.9083	0.0596	7
	24	0.2526	0.2399	0.2162	0.1694	0.3656	0.7161	0.8881	0.9566	0.0835	7
	27	0.2003	0.1890	0.1717	0.1310	0.3451	0.7206	0.8972	0.9618	0.0584	8
	30	0.2389	0.2303	0.2104	0.1675	0.4196	0.7963	0.9389	0.9817	0.0752	8
	33	0.2781	0.2655	0.2473	0.2007	0.4905	0.8546	0.9647	0.9909	0.0947	8
	36	0.2235	0.2134	0.1985	0.1554	0.4597	0.8492	0.9656	0.9922	0.0634	9

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
5	25	0.1570	0.1537	0.1478	0.1317	0.2926	0.6266	0.8348	0.9321	0.1003	5
	30	0.1042	0.1009	0.0965	0.0859	0.2670	0.6382	0.8591	0.9501	0.0604	6
	35	0.1399	0.1365	0.1309	0.1173	0.3525	0.7493	0.9262	0.9801	0.0858	6
	40	0.0937	0.0916	0.0866	0.0783	0.3233	0.7548	0.9354	0.9847	0.0526	7
	45	0.1193	0.1157	0.1113	0.1014	0.4020	0.8291	0.9663	0.9943	0.0685	7
	50	0.1471	0.1428	0.1380	0.1246	0.4721	0.8847	0.9835	0.9976	0.0860	7
	55	0.1739	0.1694	0.1637	0.1500	0.5401	0.9204	0.9914	0.9992	0.1057	7

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
7	33	0.1019	0.1016	0.0977	0.0904	0.2380	0.5875	0.8261	0.9365	0.0744	5
	40	0.0621	0.0621	0.0600	0.0551	0.2165	0.6080	0.8607	0.9572	0.0431	6
	47	0.0884	0.0868	0.0842	0.0785	0.2963	0.7277	0.9300	0.9846	0.0618	6
	54	0.1164	0.1164	0.1122	0.1049	0.3790	0.8185	0.9655	0.9946	0.0845	6
	61	0.0734	0.0719	0.0691	0.0643	0.3488	0.8211	0.9719	0.9962	0.0495	7
	68	0.0926	0.0904	0.0881	0.0820	0.4208	0.8785	0.9864	0.9987	0.0630	7
	75	0.1117	0.1107	0.1063	0.1010	0.4913	0.9201	0.9935	0.9996	0.0781	7

Table 4.8: Approximate powers of Procedure 2 at the 5% significance level

(a) N = 60

k	N_0	λ									D2
		0.222	0.222	0.308	0.5	2	3	4	4.67	1	
3	18	0.0818	0.0818	0.0681	0.0401	0.2639	0.5599	0.6754	0.7034	0.0144	7
	21	0.1145	0.1145	0.0977	0.0604	0.3699	0.6558	0.7241	0.7322	0.0206	7
	24	0.1481	0.1481	0.1285	0.0831	0.4729	0.7175	0.7463	0.7422	0.0271	7
	27	0.1810	0.1810	0.1591	0.1070	0.5661	0.7543	0.7547	0.7448	0.0335	7
	30	0.1123	0.1123	0.0960	0.0592	0.5396	0.7567	0.7532	0.7420	0.0122	8
	33	0.1301	0.1301	0.1121	0.0712	0.6239	0.7748	0.7545	0.7421	0.0138	8
	36	0.1459	0.1459	0.1263	0.0821	0.6946	0.7827	0.7547	0.7421	0.0149	8

k	N_0	λ									D2
		0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1	
5	25	0.0270	0.0270	0.0270	0.0166	0.3667	0.6185	0.7263	0.7449	0.0144	5
	30	0.0343	0.0343	0.0343	0.0214	0.5025	0.7168	0.7622	0.7513	0.0175	5
	35	0.0393	0.0393	0.0393	0.0246	0.6242	0.7712	0.7722	0.7518	0.0192	5
	40	0.0415	0.0415	0.0415	0.0260	0.7214	0.7945	0.7734	0.7518	0.0197	5
	45	0.0420	0.0420	0.0420	0.0263	0.7889	0.8001	0.7735	0.7518	0.0197	5
	50	0.0420	0.0420	0.0420	0.0263	0.8239	0.8004	0.7735	0.7518	0.0197	5
	55	0.0081	0.0081	0.0081	0.0042	0.8164	0.7925	0.7671	0.7460	0.0028	6

k	N_0	λ									D2
		0	0.455	0.455	0.455	1.67	2.5	3.57	5	1	
6	29	0.0102	0.0060	0.0060	0.0060	0.1472	0.5599	0.7332	0.7414	0.0043	5
	35	0.0113	0.0066	0.0066	0.0066	0.2353	0.6907	0.7620	0.7427	0.0045	5
	41	0.0115	0.0067	0.0067	0.0067	0.3472	0.7657	0.7662	0.7427	0.0045	5
	47	0.0115	0.0067	0.0067	0.0067	0.4862	0.7929	0.7663	0.7427	0.0045	5
	53	0.0115	0.0067	0.0067	0.0067	0.6590	0.7949	0.7663	0.7427	0.0045	5
	59	0.0115	0.0067	0.0067	0.0067	0.7923	0.7949	0.7663	0.7427	0.0045	5
	65	-	-	-	-	-	-	-	-	-	-

(b) N = 420

k	N_0	λ									D2
		0.199	0.246	0.333	0.5	2	3	4	5	1	
3	18	0.1426	0.1324	0.1134	0.0799	0.2885	0.5415	0.6509	0.6927	0.0412	7
	21	0.1169	0.1081	0.0917	0.0623	0.2759	0.5505	0.6622	0.7002	0.0270	8
	24	0.1608	0.1504	0.1306	0.0927	0.3638	0.6280	0.7057	0.7232	0.0410	8
	27	0.1297	0.1206	0.1034	0.0708	0.3429	0.6284	0.7089	0.7244	0.0264	9
	30	0.1670	0.1567	0.1368	0.0974	0.4216	0.6808	0.7319	0.7338	0.0372	9
	33	0.2059	0.1945	0.1725	0.1271	0.4932	0.7178	0.7455	0.7385	0.0500	9
	36	0.1655	0.1555	0.1367	0.0987	0.4750	0.7174	0.7455	0.7373	0.0336	10

k	N_0	λ									D2
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1	
5	25	0.0436	0.0421	0.0390	0.0321	0.2475	0.5478	0.6712	0.7160	0.0257	6
	30	0.0695	0.0674	0.0630	0.0529	0.3508	0.6452	0.7255	0.7416	0.0419	6
	35	0.0455	0.0440	0.0408	0.0338	0.3342	0.6543	0.7324	0.7432	0.0244	7
	40	0.0651	0.0631	0.0589	0.0496	0.4252	0.7117	0.7552	0.7503	0.0357	7
	45	0.0420	0.0406	0.0376	0.0312	0.4062	0.7152	0.7566	0.7488	0.0206	8
	50	0.0565	0.0546	0.0510	0.0428	0.4859	0.7498	0.7662	0.7507	0.0283	8
	55	0.0728	0.0706	0.0660	0.0561	0.5568	0.7725	0.7711	0.7513	0.0373	8

k	N_0	λ									D2
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1	
7	33	0.0177	0.0168	0.0158	0.0147	0.1643	0.5029	0.6759	0.7242	0.0125	6
	40	0.0305	0.0290	0.0275	0.0258	0.2554	0.6190	0.7330	0.7473	0.0216	6
	47	0.0465	0.0444	0.0422	0.0399	0.3506	0.6963	0.7604	0.7552	0.0332	6
	54	0.0265	0.0252	0.0238	0.0223	0.3284	0.7032	0.7624	0.7530	0.0173	7
	61	0.0372	0.0354	0.0335	0.0316	0.4149	0.7478	0.7719	0.7545	0.0245	7
	68	0.0495	0.0472	0.0448	0.0424	0.4956	0.7762	0.7762	0.7549	0.0329	7
	75	0.0277	0.0262	0.0247	0.0232	0.4704	0.7764	0.7738	0.7519	0.0168	8

Table 4.9: Simulated sample size of Procedure 2 at the 5% significance level

(a) N = 60

k	N_0	λ									D2
		0.222	0.222	0.308	0.5	2	3	4	4.67	1	
3	18	17.56	17.53	17.64	17.79	17.28	15.63	14.00	13.09	17.93	7
	21	20.21	20.23	20.35	20.59	19.60	16.87	14.50	13.36	20.87	7
	24	22.78	22.76	22.95	23.33	21.62	17.64	14.73	13.44	23.81	7
	27	25.22	25.27	25.49	25.96	23.37	18.10	14.82	13.44	26.69	7
	30	28.85	28.81	28.99	29.37	26.63	20.59	16.82	15.28	29.88	8
	33	31.44	31.43	31.68	32.15	28.13	20.74	16.78	15.31	32.82	8
	36	34.02	34.04	34.28	34.85	29.39	20.81	16.85	15.34	35.76	8

k	N_0	λ									D2
		0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1	
5	25	24.46	24.48	24.48	24.63	22.99	20.24	16.89	13.85	24.77	5
	30	29.14	29.12	29.12	29.41	26.27	21.77	17.36	13.85	29.61	5
	35	33.72	33.69	33.68	34.09	28.81	22.52	17.39	13.87	34.41	5
	40	38.21	38.18	38.26	38.73	30.61	22.75	17.42	13.89	39.21	5
	45	42.76	42.81	42.77	43.42	31.74	22.85	17.40	13.87	43.97	5
	50	47.20	47.26	47.29	48.09	32.13	22.89	17.43	13.91	48.78	5
	55	54.15	54.18	54.16	54.53	37.34	26.75	20.37	16.37	54.75	6

k	N_0	λ									D2
		0	0.455	0.455	0.455	1.67	2.5	3.57	5	1	
6	29	28.62	28.76	28.76	28.74	28.18	24.66	19.30	14.64	28.86	5
	35	34.40	34.60	34.60	34.60	33.38	26.94	19.59	14.61	34.79	5
	41	40.17	40.41	40.43	40.44	38.13	28.05	19.56	14.71	40.72	5
	47	45.94	46.23	46.28	46.27	42.23	28.41	19.62	14.69	46.61	5
	53	51.67	52.07	52.12	52.10	45.32	28.49	19.54	14.63	52.57	5
	59	57.47	57.96	57.95	57.99	46.83	28.41	19.58	14.69	58.47	5
	65	-	-	-	-	-	-	-	-	-	-

(b) N = 420

k	N_0	λ									D2
		0.199	0.246	0.333	0.5	2	3	4	5	1	
3	18	17.29	17.37	17.47	17.64	17.11	15.52	14.00	12.76	17.84	7
	21	20.39	20.47	20.53	20.70	20.06	18.17	16.21	14.72	20.89	8
	24	22.97	23.06	23.19	23.46	22.36	19.38	16.82	14.97	23.79	8
	27	26.16	26.22	26.34	26.57	25.40	22.02	18.93	16.84	26.85	9
	30	28.73	28.85	29.01	29.30	27.49	22.93	19.32	16.99	29.78	9
	33	31.23	31.37	31.58	32.00	29.37	23.55	19.47	16.99	32.65	9
	36	34.63	34.66	34.85	35.22	32.55	26.20	21.56	18.84	35.77	10

k	N_0	λ									D2
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1	
5	25	24.57	24.56	24.62	24.67	23.78	21.04	18.33	15.89	24.78	6
	30	29.17	29.18	29.25	29.35	27.68	23.10	19.27	16.25	29.59	6
	35	34.44	34.48	34.48	34.57	32.71	27.14	22.42	18.80	34.73	7
	40	39.10	39.12	39.15	39.29	36.24	28.52	22.79	18.93	39.54	7
	45	44.40	44.42	44.46	44.57	41.25	32.39	25.85	21.42	44.73	8
	50	49.12	49.13	49.18	49.33	44.62	33.28	25.97	21.39	49.61	8
	55	53.74	53.77	53.88	54.06	47.43	33.80	26.01	21.36	54.44	8

k	N_0	λ									D2
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1	
7	33	32.71	32.72	32.74	32.77	32.00	28.47	23.93	20.20	32.83	6
	40	39.43	39.47	39.45	39.51	37.92	31.82	25.09	20.55	39.64	6
	47	46.01	46.04	46.06	46.15	43.29	33.99	25.65	20.73	46.37	6
	54	53.41	53.41	53.46	53.49	50.53	39.43	29.50	23.69	53.69	7
	61	60.11	60.15	60.17	60.22	55.58	40.88	29.74	23.63	60.42	7
	68	66.71	66.72	66.71	66.84	60.14	41.70	29.88	23.71	67.20	7
	75	74.21	74.29	74.32	74.35	67.54	47.10	33.46	26.73	74.57	8

(c) N = 1680

k	N_0	λ									D2
		0.199	0.249	0.333	0.5	2	3	4	5	1	
3	18	17.27	17.34	17.45	17.62	17.09	15.53	14.00	12.79	17.84	7
	21	20.36	20.42	20.52	20.66	20.04	18.13	16.23	14.73	20.87	8
	24	22.92	23.00	23.16	23.38	22.32	19.42	16.82	15.03	23.76	8
	27	26.10	26.16	26.29	26.53	25.32	21.96	19.01	16.95	26.84	9
	30	28.65	28.79	28.92	29.24	27.45	22.89	19.35	16.97	29.73	9
	33	31.90	31.99	32.13	32.42	30.49	25.55	21.47	18.87	32.82	10
	36	34.46	34.64	34.75	35.12	32.46	26.17	21.60	18.94	35.72	10

k	N_0	λ									D2
		0.2	0.253	0.33	0.504	2	3	4	4.98	1	
5	25	24.52	24.55	24.55	24.62	23.74	21.02	18.14	15.84	24.75	6
	30	29.65	29.66	29.69	29.73	28.71	25.22	21.43	18.54	29.84	7
	35	34.35	34.37	34.41	34.49	32.59	27.25	22.19	18.76	34.68	7
	40	39.56	39.56	39.58	39.66	37.55	31.17	25.29	21.32	39.80	8
	45	44.28	44.28	44.33	44.43	41.24	32.54	25.67	21.40	44.68	8
	50	48.91	48.99	49.01	49.16	44.47	33.36	25.76	21.42	49.48	8
	55	53.49	53.54	53.61	53.82	47.49	34.06	25.89	21.53	54.27	8

k	N_0	λ									D2
		0.199	0.245	0.34	0.512	2	2.98	4	4.98	1	
7	33	32.66	32.68	32.69	32.72	31.78	28.36	24.03	20.60	32.80	6
	40	39.31	39.32	39.36	39.40	37.52	31.57	25.24	21.00	39.57	6
	47	46.53	46.56	46.56	46.62	44.46	37.14	29.30	24.24	46.72	7
	54	53.24	53.26	53.27	53.33	49.75	39.23	29.83	24.32	53.52	7
	61	59.83	59.88	59.87	59.97	54.45	40.76	30.19	24.34	60.24	7
	68	67.28	67.30	67.29	67.36	61.69	45.87	34.01	27.43	67.54	8
	75	73.90	73.97	73.98	74.12	66.08	46.92	34.02	27.38	74.35	8

(d) N = 1680000

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
3	18	17.27	17.35	17.45	17.61	17.07	15.50	14.01	12.79	17.82	7
	21	20.34	20.40	20.51	20.66	20.03	18.13	16.23	14.74	20.87	8
	24	22.92	23.00	23.13	23.40	22.29	19.37	16.85	15.02	23.77	8
	27	26.07	26.14	26.27	26.50	25.34	22.00	18.98	16.95	26.84	9
	30	28.62	28.72	28.89	29.21	27.42	22.94	19.36	17.03	29.72	9
	33	31.86	31.96	32.13	32.39	30.51	25.51	21.51	18.89	32.81	10
	36	34.45	34.57	34.72	35.08	32.45	26.13	21.70	18.90	35.71	10

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
5	25	24.52	24.50	24.55	24.62	23.77	21.07	18.19	15.83	24.74	6
	30	29.63	29.65	29.66	29.71	28.67	25.23	21.46	18.54	29.81	7
	35	34.32	34.35	34.38	34.47	32.60	27.22	22.20	18.79	34.66	7
	40	39.51	39.53	39.58	39.63	37.57	31.27	25.28	21.27	39.76	8
	45	44.24	44.26	44.32	44.39	41.15	32.61	25.60	21.35	44.63	8
	50	49.45	49.48	49.52	49.58	46.23	36.58	28.72	23.92	49.76	9
	55	54.20	54.23	54.27	54.35	49.58	37.52	28.87	23.91	54.65	9

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
7	33	32.65	32.65	32.68	32.70	31.74	28.22	24.07	20.53	32.78	6
	40	39.29	39.26	39.31	39.38	37.52	31.44	25.34	21.01	39.55	6
	47	46.52	46.54	46.53	46.57	44.47	36.88	29.41	24.17	46.71	7
	54	53.16	53.18	53.23	53.30	49.72	39.06	30.00	24.29	53.44	7
	61	60.44	60.50	60.49	60.54	56.71	44.56	33.97	27.40	60.66	8
	68	67.17	67.18	67.22	67.29	61.70	45.83	34.18	27.40	67.51	8
	75	73.78	73.82	73.86	73.94	66.17	46.94	34.32	27.47	74.25	8

Table 4.10: Simulated sample size of Procedure 2 at the 10% significance level

(a) N = 60

k	N_0	λ									D2
		0.222	0.222	0.308	0.5	2	3	4	4.67	1	
3	18	16.93	16.95	17.09	17.41	16.54	14.28	12.40	11.44	17.75	6
	21	19.29	19.27	19.53	19.98	18.47	15.05	12.66	11.60	20.58	6
	24	21.49	21.48	21.77	22.44	20.07	15.48	12.79	11.59	23.36	6
	27	25.24	25.21	25.49	25.98	23.36	18.09	14.84	13.45	26.68	7
	30	27.64	27.62	27.93	28.58	24.79	18.30	14.81	13.49	29.56	7
	33	30.02	29.94	30.26	31.09	25.96	18.41	14.82	13.44	32.44	7
	36	32.12	32.18	32.56	33.55	26.73	18.36	14.84	13.47	35.25	7

k	N_0	λ									D2
		0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1	
5	25	24.47	24.47	24.47	24.63	23.01	20.23	16.92	13.83	24.76	5
	30	29.11	29.11	29.14	29.39	26.28	21.84	17.33	13.88	29.61	5
	35	33.71	33.67	33.70	34.11	28.83	22.56	17.38	13.88	34.42	5
	40	38.19	38.22	38.20	38.77	30.57	22.80	17.37	13.87	39.25	5
	45	42.72	42.73	42.75	43.38	31.63	22.99	17.39	13.84	43.98	5
	50	47.26	47.18	47.21	48.01	32.15	22.86	17.46	13.88	48.79	5
	55	51.75	51.79	51.77	52.66	32.14	22.84	17.39	13.86	53.64	5

k	N_0	λ									D2
		0	0.455	0.455	0.455	1.67	2.5	3.57	5	1	
6	29	27.28	27.67	27.72	27.68	26.66	21.74	16.05	12.00	28.14	4
	35	32.42	32.97	33.00	33.03	30.89	23.00	16.15	12.01	33.61	4
	41	37.41	38.17	38.18	38.14	34.40	23.49	16.22	12.02	39.09	4
	47	42.29	43.39	43.35	43.35	37.32	23.59	16.21	12.00	44.69	4
	53	47.15	48.54	48.58	48.60	38.77	23.57	16.21	12.02	50.24	4
	59	52.17	53.74	53.65	53.79	39.30	23.57	16.18	12.02	55.67	4
	65	-	-	-	-	-	-	-	-	-	-

(b) N = 420

k	N_0	λ									D2
		0.199	0.246	0.333	0.5	2	3	4	5	1	
3	18	16.57	16.66	16.83	17.15	16.33	14.22	12.48	11.20	17.56	6
	21	19.77	19.87	20.04	20.30	19.37	16.80	14.70	13.10	20.72	7
	24	22.10	22.25	22.47	22.88	21.30	17.79	15.00	13.24	23.50	7
	27	25.45	25.58	25.77	26.11	24.39	20.31	17.15	15.13	26.65	8
	30	27.81	27.97	28.20	28.66	26.22	20.98	17.39	15.18	29.49	8
	33	30.05	30.27	30.55	31.16	27.83	21.46	17.44	15.16	32.25	8
	36	32.19	32.38	32.83	33.54	29.17	21.72	17.53	15.16	34.97	8

k	N_0	λ									D2
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1	
5	25	23.79	23.83	23.89	24.04	22.61	19.09	16.08	13.63	24.29	5
	30	29.18	29.19	29.26	29.35	27.66	23.11	19.26	16.25	29.57	6
	35	33.62	33.63	33.72	33.90	31.07	24.49	19.72	16.34	34.28	6
	40	37.90	37.92	38.04	38.34	33.97	25.45	19.95	16.38	38.88	6
	45	43.62	43.67	43.78	43.93	39.33	29.39	23.06	18.89	44.31	7
	50	48.06	48.08	48.24	48.49	42.04	29.94	23.06	18.95	49.02	7
	55	52.32	52.45	52.61	52.88	44.46	30.32	23.04	18.89	53.65	7

k	N_0	λ									D2
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1	
7	33	32.07	32.09	32.14	32.18	30.87	26.13	21.04	17.39	32.38	5
	40	38.32	38.37	38.39	38.48	36.07	28.46	21.68	17.61	38.83	5
	47	46.03	46.06	46.09	46.13	43.31	34.04	25.66	20.71	46.36	6
	54	52.43	52.50	52.55	52.62	48.18	35.41	25.94	20.70	53.00	6
	61	58.71	58.76	58.80	58.93	52.40	36.42	26.03	20.69	59.53	6
	68	66.66	66.75	66.74	66.82	60.01	41.59	29.82	23.73	67.22	7
	75	73.20	73.23	73.34	73.42	64.05	42.20	29.81	23.74	73.86	7

(c) N = 1680

k	N_0	λ									D2
		0.199	0.249	0.333	0.5	2	3	4	5	1	
3	18	17.27	17.35	17.44	17.62	17.08	15.52	14.00	12.78	17.82	7
	21	19.70	19.82	20.00	20.27	19.29	16.84	14.66	13.09	20.68	7
	24	22.00	22.17	22.43	22.82	21.29	17.75	15.06	13.27	23.45	7
	27	25.37	25.49	25.70	26.04	24.36	20.33	17.20	15.15	26.62	8
	30	27.72	27.82	28.09	28.60	26.23	21.01	17.40	15.22	29.42	8
	33	31.10	31.25	31.46	31.90	29.29	23.61	19.54	17.08	32.61	9
	36	33.52	33.59	33.94	34.47	31.04	24.00	19.65	17.07	35.44	9

k	N_0	λ									D2
		0.2	0.253	0.33	0.504	2	3	4	4.98	1	
5	25	24.51	24.54	24.56	24.64	23.76	21.07	18.22	15.84	24.75	6
	30	29.10	29.09	29.15	29.26	27.62	23.12	19.12	16.20	29.50	6
	35	33.47	33.51	33.58	33.79	31.03	24.54	19.55	16.35	34.14	6
	40	38.95	38.95	39.02	39.16	36.06	28.65	22.58	18.90	39.43	7
	45	43.44	43.50	43.60	43.73	39.31	29.70	22.83	18.94	44.19	7
	50	47.73	47.83	47.92	48.15	42.08	30.17	22.93	18.93	48.77	7
	55	53.48	53.57	53.59	53.80	47.38	34.18	25.80	21.41	54.24	8

k	N_0	λ									D2
		0.199	0.245	0.34	0.512	2	2.98	4	4.98	1	
7	33	31.97	31.97	32.03	32.10	30.50	25.91	21.13	17.72	32.28	5
	40	39.32	39.33	39.36	39.43	37.52	31.59	25.30	20.95	39.57	6
	47	45.83	45.84	45.86	45.94	42.63	33.85	25.99	21.22	46.20	6
	54	52.11	52.20	52.23	52.38	47.17	35.20	26.22	21.21	52.72	6
	61	59.83	59.83	59.91	59.97	54.53	40.70	30.14	24.32	60.26	7
	68	66.24	66.35	66.41	66.52	58.81	41.68	30.23	24.33	66.87	7
	75	72.64	72.61	72.73	72.90	62.57	42.35	30.31	24.44	73.47	7

(d) N = 1680000

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
3	18	17.27	17.35	17.44	17.60	17.06	15.50	13.98	12.81	17.83	7
	21	19.72	19.81	19.97	20.26	19.32	16.82	14.66	13.13	20.66	7
	24	22.00	22.15	22.36	22.80	21.29	17.73	15.09	13.30	23.43	7
	27	25.36	25.47	25.67	26.04	24.36	20.34	17.23	15.14	26.61	8
	30	27.67	27.81	28.06	28.57	26.14	21.01	17.47	15.24	29.38	8
	33	29.92	30.05	30.40	31.02	27.79	21.52	17.52	15.24	32.13	8
	36	33.44	33.58	33.91	34.40	30.97	24.19	19.70	17.15	35.41	9

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
5	25	23.70	23.70	23.77	23.94	22.56	19.09	15.91	13.61	24.18	5
	30	29.08	29.10	29.14	29.26	27.63	23.15	19.09	16.23	29.49	6
	35	33.44	33.49	33.55	33.75	30.99	24.59	19.58	16.32	34.13	6
	40	38.89	38.94	39.00	39.14	36.08	28.60	22.62	18.89	39.41	7
	45	43.37	43.38	43.49	43.68	39.31	29.53	22.87	18.99	44.10	7
	50	47.70	47.75	47.88	48.09	42.06	30.22	22.97	18.96	48.72	7
	55	51.84	51.97	52.04	52.38	44.44	30.78	22.94	18.94	53.26	7

k	N_0	λ									D2
		0.2	0.25	0.333	0.5	2	3	4	5	1	
7	33	31.93	31.94	32.00	32.07	30.48	25.88	21.17	17.71	32.25	5
	40	39.31	39.33	39.32	39.38	37.45	31.44	25.38	21.00	39.55	6
	47	45.73	45.77	45.75	45.86	42.55	33.68	26.05	21.20	46.14	6
	54	52.01	52.01	52.09	52.25	47.19	35.17	26.37	21.30	52.63	6
	61	59.73	59.74	59.83	59.91	54.48	40.63	30.33	24.29	60.16	7
	68	66.17	66.25	66.23	66.35	58.80	41.70	30.43	24.43	66.79	7
	75	72.44	72.49	72.54	72.68	62.44	42.33	30.59	24.41	73.23	7

5 Power of the χ^2-test, Approximation and Fast Algorithm

5.1 Power of the χ^2-test

Because a large value of the overall discrepancy indicates the rejection of the null hypothesis, the upper tail of the χ^2 distribution $\chi^2_{k-1,\alpha}$ constitutes the rejection region. The powers of the inverse sampling procedures can be compared with those of the χ^2-test to check the effectiveness of the procedures. The power of the standard χ^2-test is obtained by the following formula:

$$\beta_{\chi^2_{k-1,\alpha}}(n) = \sum_{\substack{x_1+x_2+...+x_k=n \\ x_1\le M_1, x_2\le M_2,...,x_k\le M_k \\ T=\frac{k}{n}\sum_{i=1}^{k}(x_i-\frac{n}{k})^2 \\ T\ge\chi^2_{k-1,\alpha}}} \cdots \sum \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}\cdots\binom{M_k}{x_k}}{\binom{N}{n}},$$

5.2 Approximate Powers of the χ^2-test

Comparing the exact powers of the χ^2-test in Table 2.1 with the approximate results in Table 12 of Panchapakesan et al. (1998), we can see that the approximations are consistent with the exact results generally. But the smaller n is, the bigger the error of approximations are. Therefore the approximations for small n are not very accurate. When n becomes bigger and bigger the two results become closer and closer. Therefore the approximations are comparable and reasonable as the replacements of the exact results if the exact results require a heavy computational effort and time even when our fastest summation algorithm is used.

Patnaik (1949) developed a formula to approximate the power of the χ^2-test in terms of the non-central χ^2 distribution:

Let $\xi_1, \xi_2, ..., \xi_k$ be k independent normal random variables with mean $a_1, a_2, ..., a_k$, respectively. A non-central χ^2 random variable X is defined by $X = \sum_{i=1}^{k} \xi_i^2$ with $\lambda = \sum_{i=1}^{k} a_i^2$, and is denoted by $\chi^2(k, \lambda)$. The pdf of a non-central chi-square random variable, $\chi^2(k, \lambda)$, is given

by:

$$p(x;k,\lambda) = \frac{e^{-\frac{1}{2}x}e^{-\frac{1}{2}\lambda}}{2^{\frac{1}{2}k}} \sum_{j=0}^{\infty} \frac{x^{\frac{1}{2}k+j-1}\lambda^j}{\Gamma(\frac{1}{2}k+j)2^{2j}j!}$$

He showed that under the linear constraints

$$\sum_{i=1}^{k} c_i\xi_i = \rho, \sum_{i=1}^{k} c_i^2 = 1,$$

that $(\sum_{i=1}^{k} \xi_i^2 - \rho^2)$ follows a $\chi^2(k-1,\lambda)$ distribution with non-centrality parameter

$$\lambda = \sum_{i=1}^{k} a_i^2 - (\sum_{i=1}^{k} a_i c_i)^2.$$

For our application, we use $T = \frac{k}{n}\sum_{i=1}^{k}(X_i - \frac{n}{k})^2$ to measure the overall discrepancy under the null hypothesis, where n is the sample size and k is the number of categories. Note that under the null hypothesis $E(X_i) = \frac{n}{k}$, for $i = 1, 2, ..., k$. We can approximate X_i by $X_i = \frac{n}{k} + \epsilon_i$, where the ϵ_i's are independent normal random variables. So $X_1, X_2, ..., X_k$ can be approximated by k independent normal random variables.

Note that $T = \frac{k}{n}\sum_{i=1}^{k}(X_i - \frac{n}{k})^2 = \frac{k}{n}\sum_{i=1}^{k}(X_i^2 - 2\frac{n}{k}X_i + \frac{n^2}{k^2}) = \frac{k}{n}\sum_{i=1}^{k}(X_i^2 - 2\frac{n^2}{k} + \frac{n^2}{k^2}k) = \frac{k}{n}(\sum_{i=1}^{k} X_i^2 - \frac{n^2}{k}) = \sum_{i=1}^{k}(\sqrt{\frac{k}{n}}X_i)^2 - (\sqrt{n})^2)$. Therefore if we take $\rho = \sqrt{n}$, $\xi_i = \sqrt{\frac{k}{n}}X_i$, and the above linear constraints are satisfied, then T will follow an approximate $\chi^2(k-1,\lambda)$ distribution. In order to check the constraints, let

$$\rho = \sqrt{n}, \;\; \xi_i = \sqrt{\frac{k}{n}}X_i,$$

and

$$c_i = \frac{1}{\sqrt{k}}, \;\; i = 1, 2, ..., k.$$

Then

$$\sum_{i=1}^{k} c_i^2 = \sum_{i=1}^{k}(\frac{1}{\sqrt{k}})^2 = 1$$

and

$$\sum_{i=1}^{k} c_i\xi_i = \frac{1}{\sqrt{k}}\sum_{i=1}^{k}\sqrt{\frac{k}{n}}X_i = \frac{\sum_{i=1}^{k} X_i}{\sqrt{n}} = \sqrt{n} = \rho.$$

Let

$$E(X_i) = n_i = n\frac{M_i}{N}, \;\; i = 1, 2, ..., k.$$

Then

$$a_i = E(\xi_i) = E(\sqrt{\frac{k}{n}}X_i) = \sqrt{\frac{k}{n}}n_i.$$

Therefore

$$\lambda = \sum_{i=1}^{k} a_i^2 - (\sum_{i=1}^{k} a_i c_i)^2 = \sum_{i=1}^{k}(\sqrt{\frac{k}{n}}n_i)^2 - (\sum_{i=1}^{k}\sqrt{\frac{k}{n}}n_i\frac{1}{\sqrt{k}})^2 = \frac{k}{n}\sum_{i=1}^{k} n_i^2 - n.$$

Therefore T follows an approximate non-central χ^2 distribution with $(k-1)$ degrees of freedom and non-centrality parameter

$$\lambda = \frac{\sum_{i=1}^{2} n_i^2}{\frac{n}{k}} - n. \qquad (5.1)$$

Under the alternative hypothesis H_a, $n_1 = n_2 = ... = n_{k-1} = n\frac{M_1}{N}$ and $n_k = n\frac{M_k}{N}$. Therefore the power of the χ^2-test can be approximated by

$$\beta_{\chi^2_{k-1,\alpha}}(n, \lambda) \approx \int_{\chi^2_{k-1,\alpha}}^{\infty} p(x; k-1, \lambda)dx,$$

where λ is given in equation (5.1).

5.3 Results for Powers of χ^2-test

The significance levels of the χ^2-test are 3% and 7%, for providing consistency with the 5% and 10% significance levels of the two procedures, respectively that under the null hypothesis ($\lambda = 1$) the powers of the procedures are close to the powers of χ^2-test for all values of k. In tables 5.1 and 5.2, we present the powers of the χ^2-test at the 3% and 7% significance level,

respectively, for $k = 3(2)7$ λ's are selected to be as close as possible to $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 2, 3, 4, 5, 1$. From Tables 5.1 and 5.2, we see that under the same n and λ, the bigger k is, the smaller the powers of the χ^2-test are and the powers of the χ^2-test decrease more quickly as k increases when λ decreases. For example, under $N = 420$ and $n = 30$, when $\lambda = 3$, the powers of the χ^2-test are 0.7425 and 0.5583 for $k = 3$ and $k = 5$, respectively. The difference of the decrease is $0.7425 - 0.5583 = 0.1842$. But when $\lambda = 0.2$, the powers of the χ^2-test are 0.7769 and 0.1734 for $k = 3$ and $k = 5$, respectively. The difference of the decrease is $0.7769 - 0.1734 = 0.6035$. Therefore for $\lambda < 1$, it needs bigger sample size for the procedures as k increases. When N is a small number like $N = 60$, for $\lambda < 1$, the powers of the χ^2-test are becoming too small as k increases since for big k we use big n. And big n, small λ and $x_i < M_i$ for $i = 1, 2, ..., k$ make x_i's close to n/k and hence $T < \chi^2_{k-1,\alpha}$. Therefore for small N like $N = 60$ when $\lambda < 1$, the χ^2-test should not be used. Comparing Tables 4.1 and 4.4 with Table 5.1, we see that the expected sample sizes of χ^2-test are bigger than those of Procedure 1 when the powers of the χ^2-test are close to those of Procedure 1 under the same N, k, and λ. For example, $N = 1680, k = 3$, and $\lambda = 4$, from Table 4.1 we find that the power equals 0.8650 corresponding $M = 15$. From Table 4.4 at the same N, k, λ and M we find that the sample size is 22.49. From Table 5.1 at the same N, k, λ we find the power equals 0.8600 and find corresponding $n = 27$. Therefore the sample size of the χ^2-test, $n = 27$, is greater than that of Procedure 1. Comparing Table 5.1 with Tables 4.6 and 4.9, we draw the similar conclusion that the expected sample sizes of χ^2-test are bigger than those of Procedure 2 when the powers of the χ^2-test are close to those of Procedure 2 under the same N, k, and λ. When $N \geq 1680000$ the results of Tables 5.1 and 5.2 are the same as Tables 12 and 13 in the paper of Panchapakesan et al. (1998).

Table 5.1: Powers of the χ^2-test at the 3% significance level

(a) N = 60

k	n	λ								
		0.222	0.222	0.308	0.5	2	3	4	4.67	1
3	18	0.1933	0.1933	0.1057	0.0294	0.0893	0.3623	0.6330	0.7604	0.0029
	21	0.4062	0.4062	0.2326	0.0630	0.1578	0.5430	0.8143	0.9060	0.0049
	24	0.3799	0.3799	0.2035	0.0442	0.1272	0.5262	0.8210	0.9164	0.0017
	27	0.4969	0.4969	0.2559	0.0456	0.1595	0.6517	0.9167	0.9732	0.0009
	30	0.7083	0.7083	0.4087	0.0761	0.2319	0.7896	0.9733	0.9950	0.0010
	33	0.7388	0.7388	0.4252	0.0687	0.2230	0.8080	0.9803	0.9970	0.0004
	36	0.8314	0.8314	0.4891	0.0674	0.2417	0.8766	0.9952	0.9998	0.0001

k	n	λ								
		0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1
5	25	0.0210	0.0210	0.0210	0.0033	0.0732	0.2817	0.5996	0.8594	0.0007
	30	0.0141	0.0141	0.0141	0.0010	0.0724	0.3390	0.7120	0.9359	0.0001
	35	0.0122	0.0122	0.0122	0.0003	0.0720	0.4195	0.8382	0.9864	0.0000
	40	0.0126	0.0126	0.0126	0.0001	0.0780	0.5152	0.9154	0.9970	0.0000
	45	0.0059	0.0059	0.0059	0.0000	0.0580	0.6244	0.9823	1.0000	0.0000
	50	0.0045	0.0045	0.0045	0.0000	0.0380	0.6711	0.9969	1.0000	0.0000
	55	0.0011	0.0011	0.0011	0.0000	0.0002	0.9222	1.0000	1.0000	0.0000

k	n	λ								
		0	0.455	0.455	0.455	1.67	2.5	3.57	5	1
6	29	0.0278	0.0013	0.0013	0.0013	0.0091	0.1799	0.6349	0.9413	0.0001
	35	0.0300	0.0003	0.0003	0.0003	0.0047	0.2028	0.7446	0.9830	0.0000
	41	0.0256	0.0000	0.0000	0.0000	0.0014	0.2684	0.8795	0.9980	0.0000
	47	0.0141	0.0000	0.0000	0.0000	0.0000	0.2981	0.9746	1.0000	0.0000
	53	0.0121	0.0000	0.0000	0.0000	0.0000	0.2527	0.9988	1.0000	0.0000
	59	0.0000	0.0000	0.0000	0.0000	0.0000	0.6667	1.0000	1.0000	0.0000
	65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

(b) N = 420

k	n	λ								
		0.199	0.246	0.333	0.5	2	3	4	5	1
3	18	0.3052	0.2343	0.1462	0.0640	0.1310	0.3955	0.6307	0.7848	0.0140
	21	0.5366	0.4306	0.2818	0.1258	0.2165	0.5562	0.7860	0.9012	0.0264
	24	0.5287	0.4190	0.2652	0.1080	0.1969	0.5513	0.7934	0.9104	0.0170
	27	0.6316	0.5056	0.3191	0.1251	0.2370	0.6372	0.8644	0.9527	0.0172
	30	0.7769	0.6568	0.4471	0.1880	0.3170	0.7425	0.9258	0.9800	0.0256
	33	0.8015	0.6840	0.4691	0.1931	0.3250	0.7635	0.9385	0.9850	0.0221
	36	0.8525	0.7420	0.5210	0.2143	0.3563	0.8071	0.9590	0.9920	0.0213

k	n	λ								
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1
5	25	0.1444	0.1253	0.0944	0.0538	0.1581	0.4822	0.7313	0.8826	0.0227
	30	0.1734	0.1458	0.1033	0.0528	0.1788	0.5583	0.8104	0.9347	0.0186
	35	0.2364	0.1938	0.1304	0.0599	0.2102	0.6395	0.8764	0.9678	0.0176
	40	0.3325	0.2694	0.1761	0.0754	0.2570	0.7232	0.9262	0.9857	0.0194
	45	0.3784	0.3024	0.1906	0.0741	0.2757	0.7706	0.9515	0.9930	0.0156
	50	0.4574	0.3646	0.2256	0.0824	0.3117	0.8226	0.9711	0.9970	0.0150
	55	0.5424	0.4382	0.2736	0.0962	0.3480	0.8640	0.9831	0.9987	0.0149

k	n	λ								
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1
7	33	0.0907	0.0689	0.0527	0.0407	0.1147	0.4493	0.7647	0.9168	0.0171
	40	0.1128	0.0813	0.0591	0.0433	0.1346	0.5354	0.8483	0.9612	0.0150
	47	0.1458	0.1004	0.0695	0.0485	0.1593	0.6192	0.9073	0.9830	0.0140
	54	0.1811	0.1186	0.0780	0.0518	0.1836	0.6958	0.9473	0.9935	0.0122
	61	0.2297	0.1440	0.0904	0.0571	0.2091	0.7600	0.9708	0.9976	0.0111
	68	0.2827	0.1706	0.1024	0.0617	0.2347	0.8139	0.9844	0.9992	0.0097
	75	0.3463	0.2027	0.1168	0.0671	0.2611	0.8590	0.9922	0.9997	0.0085

(c) N = 1680

k	n	λ								
		0.199	0.249	0.333	0.5	2	3	4	5	1
3	18	0.3136	0.2385	0.1532	0.0686	0.1359	0.3990	0.6308	0.7827	0.0161
	21	0.5432	0.4327	0.2910	0.1335	0.2229	0.5578	0.7836	0.8980	0.0302
	24	0.5363	0.4220	0.2754	0.1161	0.2042	0.5537	0.7911	0.9072	0.0202
	27	0.6363	0.5062	0.3303	0.1350	0.2448	0.6366	0.8600	0.9490	0.0209
	30	0.7765	0.6530	0.4570	0.2003	0.3249	0.7395	0.9212	0.9773	0.0312
	33	0.8007	0.6796	0.4792	0.2064	0.3338	0.7604	0.9340	0.9828	0.0277
	36	0.8496	0.7356	0.5302	0.2291	0.3654	0.8026	0.9547	0.9902	0.0273

k	n	λ								
		0.2	0.253	0.33	0.504	2	3	4	4.98	1
5	25	0.1597	0.1350	0.1066	0.0648	0.1685	0.4891	0.7473	0.8854	0.0281
	30	0.1934	0.1578	0.1189	0.0659	0.1911	0.5637	0.8225	0.9352	0.0244
	35	0.2623	0.2082	0.1509	0.0767	0.2250	0.6429	0.8842	0.9670	0.0244
	40	0.3635	0.2853	0.2028	0.0979	0.2744	0.7243	0.9305	0.9848	0.0278
	45	0.4125	0.3195	0.2212	0.0994	0.2947	0.7694	0.9536	0.9920	0.0241
	50	0.4919	0.3810	0.2611	0.1126	0.3322	0.8193	0.9717	0.9963	0.0243
	55	0.5757	0.4534	0.3138	0.1325	0.3705	0.8594	0.9830	0.9983	0.0255

k	n	λ								
		0.199	0.245	0.34	0.512	2	2.98	4	4.98	1
7	33	0.1651	0.1437	0.1080	0.0653	0.1541	0.5441	0.8686	0.9738	0.0300
	40	0.2037	0.1759	0.1295	0.0742	0.1894	0.6519	0.9351	0.9924	0.0300
	47	0.2443	0.2101	0.1524	0.0837	0.2268	0.7421	0.9700	0.9980	0.0300
	54	0.2866	0.2459	0.1765	0.0936	0.2657	0.8141	0.9868	0.9995	0.0300
	61	0.3297	0.2828	0.2018	0.1039	0.3058	0.8693	0.9945	0.9999	0.0300
	68	0.3732	0.3205	0.2280	0.1147	0.3464	0.9100	0.9978	1.0000	0.0300
	75	0.4166	0.3585	0.2549	0.1259	0.3871	0.9394	0.9991	1.0000	0.0300

(d) N = 1680000

k	n	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
3	18	0.3145	0.2398	0.1555	0.0702	0.1375	0.4001	0.6308	0.7820	0.0168
	21	0.5428	0.4334	0.2940	0.1361	0.2250	0.5583	0.7828	0.8970	0.0315
	24	0.5362	0.4229	0.2788	0.1187	0.2066	0.5545	0.7904	0.9062	0.0213
	27	0.6350	0.5064	0.3339	0.1382	0.2473	0.6365	0.8585	0.9479	0.0222
	30	0.7740	0.6518	0.4601	0.2044	0.3275	0.7386	0.9197	0.9764	0.0331
	33	0.7980	0.6782	0.4823	0.2107	0.3366	0.7594	0.9327	0.9820	0.0296
	36	0.8465	0.7336	0.5330	0.2338	0.3683	0.8012	0.9533	0.9896	0.0294

k	n	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
5	25	0.1648	0.1410	0.1097	0.0686	0.1719	0.4913	0.7475	0.8863	0.0301
	30	0.2001	0.1658	0.1227	0.0706	0.1952	0.5655	0.8220	0.9354	0.0266
	35	0.2708	0.2189	0.1556	0.0828	0.2299	0.6441	0.8831	0.9668	0.0269
	40	0.3734	0.2991	0.2085	0.1059	0.2800	0.7247	0.9293	0.9845	0.0310
	45	0.4232	0.3352	0.2276	0.1084	0.3009	0.7692	0.9522	0.9918	0.0273
	50	0.5026	0.3985	0.2682	0.1232	0.3388	0.8184	0.9705	0.9960	0.0280
	55	0.5858	0.4720	0.3213	0.1453	0.3777	0.8582	0.9820	0.9982	0.0297

k	n	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
7	33	0.1110	0.0971	0.0784	0.0530	0.1545	0.4876	0.7665	0.9091	0.0273
	40	0.1405	0.1202	0.0936	0.0592	0.1852	0.5754	0.8469	0.9546	0.0269
	47	0.1823	0.1532	0.1157	0.0689	0.2218	0.6581	0.9040	0.9785	0.0278
	54	0.2256	0.1861	0.1364	0.0766	0.2571	0.7293	0.9424	0.9906	0.0276
	61	0.2817	0.2291	0.1637	0.0872	0.2940	0.7880	0.9658	0.9959	0.0282
	68	0.3398	0.2734	0.1913	0.0972	0.3299	0.8355	0.9800	0.9983	0.0283
	75	0.4051	0.3238	0.2228	0.1085	0.3659	0.8740	0.9887	0.9993	0.0284

Table 5.2: Powers of the χ^2-test at the 7% significance level

(a) N = 60

k	n	λ								
		0.222	0.222	0.308	0.5	2	3	4	4.67	1
3	18	0.5517	0.5517	0.3728	0.1452	0.2450	0.6118	0.8355	0.9111	0.0256
	21	0.6950	0.6950	0.4782	0.1772	0.3106	0.7356	0.9248	0.9699	0.0236
	24	0.6276	0.6276	0.3908	0.1095	0.2402	0.7028	0.9223	0.9717	0.0067
	27	0.8088	0.8088	0.5577	0.1671	0.3320	0.8269	0.9743	0.9940	0.0079
	30	0.9120	0.9120	0.6740	0.1996	0.4057	0.9070	0.9937	0.9992	0.0061
	33	0.9051	0.9051	0.6509	0.1655	0.3611	0.9028	0.9946	0.9995	0.0021
	36	0.9677	0.9677	0.7354	0.1637	0.4091	0.9540	0.9994	1.0000	0.0007

k	n	λ								
		0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1
5	25	0.0795	0.0795	0.0795	0.0149	0.1479	0.4319	0.7492	0.9360	0.0043
	30	0.0806	0.0806	0.0806	0.0078	0.1478	0.4833	0.8218	0.9713	0.0011
	35	0.0776	0.0776	0.0776	0.0030	0.1756	0.6204	0.9332	0.9970	0.0001
	40	0.0732	0.0732	0.0732	0.0010	0.1613	0.6653	0.9635	0.9995	0.0000
	45	0.0648	0.0648	0.0648	0.0002	0.1805	0.8205	0.9968	1.0000	0.0000
	50	0.0398	0.0398	0.0398	0.0000	0.1238	0.8562	0.9998	1.0000	0.0000
	55	0.0316	0.0316	0.0316	0.0000	0.1205	0.9922	1.0000	1.0000	0.0000

k	n	λ								
		0	0.455	0.455	0.455	1.67	2.5	3.57	5	1
6	29	0.1232	0.0082	0.0082	0.0082	0.0289	0.2988	0.7499	0.9660	0.0007
	35	0.1658	0.0028	0.0028	0.0028	0.0172	0.3507	0.8703	0.9960	0.0000
	41	0.2758	0.0007	0.0007	0.0007	0.0085	0.4019	0.9360	0.9997	0.0000
	47	0.5415	0.0001	0.0001	0.0001	0.0005	0.5529	0.9953	1.0000	0.0000
	53	1.0000	0.0000	0.0000	0.0000	0.0000	0.5718	1.0000	1.0000	0.0000
	59	1.0000	0.0000	0.0000	0.0000	0.0000	1.0000	1.0000	1.0000	0.0000
	65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

(b) N = 420

k	n	λ								
		0.199	0.246	0.333	0.5	2	3	4	5	1
3	18	0.6469	0.5498	0.3993	0.2142	0.2978	0.6200	0.8159	0.9121	0.0675
	21	0.7606	0.6606	0.4903	0.2635	0.3640	0.7162	0.8900	0.9580	0.0769
	24	0.7195	0.6080	0.4253	0.2010	0.3084	0.6880	0.8830	0.9578	0.0418
	27	0.8418	0.7450	0.5575	0.2822	0.3995	0.7878	0.9389	0.9829	0.0592
	30	0.9000	0.8167	0.6328	0.3288	0.4580	0.8455	0.9652	0.9924	0.0658
	33	0.9016	0.8179	0.6297	0.3142	0.4445	0.8478	0.9680	0.9936	0.0506
	36	0.9312	0.8563	0.6683	0.3287	0.4740	0.8804	0.9802	0.9969	0.0460

k	n	λ								
		0.2	0.242	0.33	0.516	2	3	3.92	4.94	1
5	25	0.3017	0.2636	0.2015	0.1194	0.2545	0.6102	0.8262	0.9348	0.0562
	30	0.3847	0.3297	0.2416	0.1305	0.2869	0.6803	0.8824	0.9650	0.0524
	35	0.4656	0.3934	0.2789	0.1399	0.3317	0.7613	0.9340	0.9860	0.0491
	40	0.5463	0.4636	0.3275	0.1575	0.3681	0.8107	0.9578	0.9930	0.0485
	45	0.6241	0.5310	0.3718	0.1695	0.4076	0.8598	0.9767	0.9973	0.0453
	50	0.6809	0.5820	0.4056	0.1763	0.4352	0.8900	0.9856	0.9987	0.0407
	55	0.7466	0.6458	0.4546	0.1928	0.4737	0.9200	0.9923	0.9995	0.0391

k	n	λ								
		0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1
7	33	0.2892	0.2320	0.1842	0.1460	0.2320	0.6622	0.9295	0.9904	0.0700
	40	0.3419	0.2715	0.2120	0.1642	0.2715	0.7549	0.9690	0.9977	0.0700
	47	0.3946	0.3117	0.2405	0.1828	0.3117	0.8269	0.9870	0.9995	0.0700
	54	0.4462	0.3521	0.2696	0.2020	0.3521	0.8807	0.9949	0.9999	0.0700
	61	0.4964	0.3923	0.2991	0.2215	0.3923	0.9195	0.9980	1.0000	0.0700
	68	0.5444	0.4321	0.3287	0.2414	0.4321	0.9468	0.9993	1.0000	0.0700
	75	0.5899	0.4710	0.3584	0.2616	0.4710	0.9654	0.9998	1.0000	0.0700

(c) N = 1680

k	n	λ								
		0.199	0.249	0.333	0.5	2	3	4	5	1
3	18	0.6507	0.5495	0.4066	0.2221	0.3037	0.6210	0.8141	0.9098	0.0735
	21	0.7613	0.6576	0.4971	0.2729	0.3699	0.7150	0.8868	0.9554	0.0846
	24	0.7216	0.6061	0.4341	0.2112	0.3154	0.6873	0.8795	0.9549	0.0478
	27	0.8401	0.7396	0.5639	0.2938	0.4062	0.7850	0.9353	0.9809	0.0678
	30	0.8962	0.8090	0.6369	0.3411	0.4637	0.8410	0.9618	0.9910	0.0761
	33	0.8982	0.8106	0.6346	0.3276	0.4516	0.8436	0.9648	0.9923	0.0603
	36	0.9264	0.8471	0.6714	0.3431	0.4800	0.8748	0.9770	0.9960	0.0562

k	n	λ								
		0.2	0.253	0.33	0.504	2	3	4	4.98	1
5	25	0.3213	0.2737	0.2190	0.1377	0.2669	0.6148	0.8370	0.9356	0.0665
	30	0.4080	0.3404	0.2637	0.1536	0.3019	0.6843	0.8906	0.9650	0.0644
	35	0.4910	0.4036	0.3052	0.1679	0.3475	0.7616	0.9377	0.9852	0.0629
	40	0.5718	0.4732	0.3577	0.1913	0.3864	0.8104	0.9600	0.9924	0.0645
	45	0.6474	0.5380	0.4049	0.2087	0.4262	0.8570	0.9773	0.9968	0.0629
	50	0.7016	0.5869	0.4411	0.2205	0.4548	0.8862	0.9857	0.9984	0.0594
	55	0.7628	0.6472	0.4913	0.2429	0.4934	0.9152	0.9920	0.9993	0.0596

k	n	λ								
		0.199	0.245	0.34	0.512	2	2.98	4	4.98	1
7	33	0.2740	0.2454	0.1953	0.1303	0.2594	0.6808	0.9295	0.9890	0.0700
	40	0.3234	0.2881	0.2258	0.1445	0.3054	0.7726	0.9690	0.9972	0.0700
	47	0.3729	0.3314	0.2571	0.1591	0.3518	0.8425	0.9870	0.9994	0.0700
	54	0.4219	0.3747	0.2889	0.1741	0.3980	0.8937	0.9949	0.9999	0.0700
	61	0.4697	0.4176	0.3211	0.1894	0.4434	0.9300	0.9980	1.0000	0.0700
	68	0.5160	0.4597	0.3533	0.2050	0.4877	0.9547	0.9993	1.0000	0.0700
	75	0.5602	0.5005	0.3854	0.2208	0.5304	0.9713	0.9998	1.0000	0.0700

(d) N = 1680000

k	n	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
3	18	0.6497	0.5493	0.4089	0.2246	0.3057	0.6214	0.8136	0.9090	0.0755
	21	0.7594	0.6566	0.4993	0.2760	0.3718	0.7147	0.8857	0.9546	0.0872
	24	0.7198	0.6055	0.4370	0.2145	0.3177	0.6870	0.8784	0.9540	0.0499
	27	0.8376	0.7379	0.5659	0.2976	0.4084	0.7842	0.9341	0.9802	0.0708
	30	0.8934	0.8066	0.6382	0.3451	0.4655	0.8395	0.9606	0.9904	0.0796
	33	0.8955	0.8083	0.6362	0.3319	0.4539	0.8423	0.9637	0.9918	0.0636
	36	0.9236	0.8442	0.6724	0.3476	0.4820	0.8730	0.9760	0.9956	0.0597

k	n	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
5	25	0.3276	0.2824	0.2226	0.1439	0.2710	0.6164	0.8366	0.9360	0.0700
	30	0.4155	0.3515	0.2679	0.1615	0.3068	0.6857	0.8900	0.9651	0.0686
	35	0.4990	0.4167	0.3099	0.1775	0.3527	0.7618	0.9367	0.9850	0.0678
	40	0.5797	0.4876	0.3629	0.2028	0.3923	0.8104	0.9592	0.9923	0.0702
	45	0.6545	0.5531	0.4101	0.2219	0.4322	0.8562	0.9764	0.9967	0.0692
	50	0.7078	0.6021	0.4463	0.2353	0.4611	0.8852	0.9850	0.9983	0.0662
	55	0.7677	0.6619	0.4963	0.2596	0.4997	0.9138	0.9914	0.9993	0.0671

k	n	λ								
		0.2	0.25	0.333	0.5	2	3	4	5	1
7	33	0.2736	0.2427	0.1983	0.1337	0.2594	0.6869	0.9295	0.9894	0.0700
	40	0.3228	0.2847	0.2295	0.1488	0.3054	0.7783	0.9690	0.9974	0.0700
	47	0.3722	0.3274	0.2616	0.1643	0.3518	0.8476	0.9870	0.9994	0.0700
	54	0.4211	0.3701	0.2942	0.1802	0.3980	0.8979	0.9949	0.9999	0.0700
	61	0.4689	0.4125	0.3270	0.1964	0.4434	0.9331	0.9980	1.0000	0.0700
	68	0.5151	0.4541	0.3599	0.2130	0.4877	0.9571	0.9993	1.0000	0.0700
	75	0.5593	0.4946	0.3926	0.2298	0.5304	0.9730	0.9998	1.0000	0.0700

6 Combination of the Two Procedures

Ideally, for given k and α, we would like to use the procedures so that the homogeneity testing in a multivariate hypergeometric distribution is optimal and quickly draws the appropriate conclusion. From the data in chapter 4 we have seen that the powers of Procedure 2 when $\lambda < 1$ are not distinctly bigger than the powers of Procedure 2 when $\lambda = 1$, the probabilities of the null hypothesis. And when $\lambda > 1$, the powers of Procedure 1 are smaller than the power of Procedure 2. Panchapakesan et al. (1998) found similar problems and proposed a combined procedure that makes use of both of the two procedures in a multinomial population and called it Procedure 3. We also used Procedure 3 for multivariate hypergeometric populations. From Chapter 3 we know that under current computer technology a single personal computer cannot do the exact calculations for Procedure 2 but simulations could be used to draw similar conclusions. Therefore we need to use simulations for Procedure 3 as well.

6.1 Procedure 3

For a given k, N, α, λ, M and N_0, we get the corresponding critical values D_1 and D_2 in Tables 3.1 and 3.2, respectively. We then take one observation at a time from the multivariate hypergeometric population with parameters related to the values of k, N, λ and previous observations in each categories until one of the following three events occurs.

(1) $X_{(k)t} - X_{(k-1)t} \geq D_2$

(2) $X_{(k)t} = M$

(3) N_0 observations have occurred

If (1) occurs, then we stop sampling and reject H_0. If (2) occurs, we perform the test in Procedure 1 and draw the conclusion. And if (3) occurs, then we stop the procedure and accept H_0. If any two, or all three, of the above events occur simultaneously, we take the action corresponding to the smallest numbered event that occurs.

6.2 Powers and Sample Size for Procedure 3

We combine Procedure 1 and 2 together so that for fixed k, λ and α the one of the above three events happens often if it has the biggest probability among all the three events under some conditions. Therefore when $\lambda < 1$, (2) would occur before (1) and (3) while (1) or (3) will occur before (2) when $\lambda > 1$.

In tables 6.1 and 6.2, we present the simulated powers of Procedure 3 at the 5% and 10% significance levels, respectively, where a missing entry "$-$" indicates that D_1 or D_2 was not attainable. The values of M and N_0 were chosen to be the same as those in Procedure 1 and 2. The percentage of times that the outcome came from Procedure 1 was presented in brackets below the powers. Looking at the percentage of times that the outcome came from Procedure 1 in Tables 6.1 and 6.2, we can immediately see that Procedure 3 works just the way we want. The percentage of times the outcome is determined from Procedure 1 increases as λ decreases. Therefore Procedure 1 occurs more often than Procedure 2 when $\lambda < 1$ while Procedure 2 occurs more often when $\lambda > 1$. The powers when $k = 3$ under $N = 60$ are distinctly larger than those under the other N's. When $k \geq 5$, under $N = 60$, the powers decrease as k increases for $\lambda < 1$. All the powers increase as N_0 and M increase. Therefore the powers show that Procedure 3 is better than the other 2 procedures.

The expected sample size of Procedure 3 at the 5% and 10% significance levels are shown in Tables 6.3 and 6.4, respectively. Comparing the expected sample size of Procedure 3 with those of both Procedure 1 and 2, we see that Procedure 3 has the advantage of having smaller expected sample size than Procedure 2 and even a little bit smaller than Procedure 1 under the same M and N_0 for all the values of k and λ. Comparing the expected sample size of Procedure 3 under all the N's we can conclude that the expected sample size are very close for all of the N's. When $N \geq 1680000$ the results of Tables 6.1, 6.2, 6.3, and 6.4 are the same as Tables 14, 15, 16, and 17 in the paper of Panchapakesan et al. (1998).

Table 6.1: Powers of Procedure 3 at the 5% significance level

(a) N = 60

k	N_0	M	λ								D2	D1
			0.222	0.308	0.5	2	3	4	4.67	1		
3	18	9	0.505	0.330	0.122	0.168	0.416	0.627	0.726	0.027	7	8
			(.75)	(.64)	(.43)	(.55)	(.66)	(.55)	(.46)	(.2)		
	21	10	0.730	0.523	0.218	0.279	0.616	0.822	0.893	0.043	7	8
			(.83)	(.73)	(.5)	(.58)	(.55)	(.36)	(.26)	(.22)		
	24	11	0.882	0.694	0.313	0.395	0.772	0.927	0.967	0.056	7	8
			(.86)	(.79)	(.55)	(.58)	(.41)	(.2)	(.12)	(.23)		
	27	12	0.851	0.634	0.256	0.400	0.805	0.951	0.982	0.040	7	9
			(.87)	(.82)	(.6)	(.55)	(.29)	(.1)	(.045)	(.24)		
	30	13	0.950	0.770	0.313	0.404	0.825	0.961	0.986	0.030	8	9
			(.93)	(.91)	(.7)	(.68)	(.36)	(.14)	(.067)	(.24)		
	33	14	1.000	0.881	0.405	0.510	0.908	0.989	0.998	0.037	8	9
			(.92)	(.92)	(.75)	(.64)	(.24)	(.059)	(.021)	(.24)		
	36	15	1.000	0.957	0.505	0.613	0.956	0.998	1.000	0.037	8	9
			(.91)	(.91)	(.78)	(.57)	(.14)	(.02)	(.005)	(.23)		

k	N_0	M	λ								D2	D1
			0.286	0.286	0.615	2	2.67	3.5	4.57	1		
5	25	9	0.167	0.167	0.066	0.312	0.610	0.842	0.959	0.038	5	8
			(.16)	(.16)	(.072)	(.24)	(.3)	(.21)	(.089)	(.039)		
	30	10	0.250	0.253	0.092	0.459	0.793	0.954	0.994	0.046	5	8
			(.2)	(.2)	(.082)	(.28)	(.24)	(.1)	(.023)	(.037)		
	35	11	0.323	0.327	0.108	0.612	0.908	0.992	1.000	0.050	5	8
			(.24)	(.24)	(.082)	(.28)	(.16)	(.032)	(0.00)	(.028)		
	40	12	0.366	0.362	0.134	0.836	0.991	1.000	1.000	0.052	5	7
			(.26)	(.26)	(.069)	(.24)	(.064)	(.00)	(.00)	(.014)		
	45	13	-	-	-	-	-	-	-	-	5	-
			-	-	-	-	-	-	-	-		
	50	14	-	-	-	-	-	-	-	-	5	-
			-	-	-	-	-	-	-	-		
	55	15	-	-	-	-	-	-	-	-	6	-
			-	-	-	-	-	-	-	-	-	-

k	N_0	M	λ								D2	D1
			0		0.455	0.455	1.67	2.5	3.57	5	1	
6	29	9	-	0.056	0.055	0.132	0.528	0.870	0.983	0.019	5	8
			(-)	(.068)	(.066)	(.16)	(.36)	(.22)	(.057)	(.021)		
	35	10	-	0.087	0.086	0.310	0.856	0.993	1.000	0.022	5	7
			(-)	(.064)	(.062)	(.21)	(.31)	(.085)	(0.00)	(.012)		
	41	11	-	-	-	-	-	-	-	-	5	-
			-	-	-	-	-	-	-	-		
	47	12	-	-	-	-	-	-	-	-	5	-
			-	-	-	-	-	-	-	-		
	53	13	-	-	-	-	-	-	-	-	5	-
			-	-	-	-	-	-	-	-		
	59	14	-	-	-	-	-	-	-	-	5	-
			-	-	-	-	-	-	-	-		
	65	15	-	-	-	-	-	-	-	-	-	-
			-	-	-	-	-	-	-	-		

(b) N = 420

k	N_0	M	λ									D2	D1
			0.199	0.246	0.333	0.5	2	3	4	5	1		
3	18	9	0.281	0.219	0.142	0.071	0.117	0.310	0.495	0.636	0.023	7	9
			(.78)	(.73)	(.63)	(.48)	(.54)	(.62)	(.54)	(.42)	(.29)		
	21	10	0.529	0.419	0.278	0.121	0.152	0.378	0.574	0.713	0.030	8	9
			(.88)	(.84)	(.75)	(.58)	(.65)	(.73)	(.63)	(.5)	(.35)		
	24	11	0.485	0.381	0.241	0.105	0.164	0.429	0.644	0.786	0.025	8	10
			(.9)	(.87)	(.79)	(.63)	(.65)	(.63)	(.46)	(.31)	(.38)		
	27	12	0.675	0.558	0.375	0.164	0.199	0.488	0.705	0.837	0.031	9	10
			(.94)	(.92)	(.87)	(.71)	(.74)	(.71)	(.53)	(.38)	(.43)		
	30	13	0.823	0.714	0.522	0.246	0.296	0.638	0.840	0.928	0.050	9	10
			(.94)	(.93)	(.88)	(.74)	(.73)	(.6)	(.38)	(.23)	(.46)		
	33	14	0.788	0.672	0.468	0.212	0.294	0.658	0.862	0.947	0.037	9	11
			(.92)	(.92)	(.89)	(.77)	(.7)	(.48)	(.26)	(.13)	(.47)		
	36	15	0.882	0.785	0.581	0.270	0.323	0.706	0.888	0.961	0.043	10	11
			(.95)	(.95)	(.93)	(.82)	(.77)	(.54)	(.31)	(.16)	(.51)		

k	N_0	M	λ									D2	D1
			0.2	0.242	0.33	0.516	2	3	3.92	4.94	1		
5	25	9	0.208	0.174	0.121	0.070	0.148	0.404	0.607	0.759	0.038	6	9
			(.4)	(.38)	(.34)	(.27)	(.42)	(.54)	(.47)	(.34)	(.19)		
	30	10	0.222	0.179	0.124	0.072	0.185	0.489	0.704	0.848	0.040	6	10
			(.5)	(.48)	(.43)	(.35)	(.47)	(.48)	(.33)	(.2)	(.24)		
	35	11	0.426	0.354	0.247	0.115	0.214	0.547	0.760	0.892	0.043	7	10
			(.64)	(.61)	(.55)	(.45)	(.62)	(.61)	(.43)	(.26)	(.31)		
	40	12	0.430	0.348	0.222	0.097	0.216	0.583	0.810	0.925	0.035	7	11
			(.72)	(.69)	(.62)	(.51)	(.63)	(.5)	(.29)	(.15)	(.35)		
	45	13	0.622	0.530	0.359	0.154	0.265	0.652	0.860	0.953	0.044	8	11
			(.8)	(.77)	(.71)	(.59)	(.74)	(.6)	(.36)	(.19)	(.4)		
	50	14	0.614	0.512	0.329	0.124	0.252	0.667	0.880	0.964	0.031	8	12
			(.85)	(.82)	(.77)	(.64)	(.73)	(.48)	(.24)	(.1)	(.44)		
	55	15	0.771	0.672	0.478	0.198	0.359	0.795	0.949	0.990	0.050	8	12
			(.88)	(.86)	(.81)	(.68)	(.7)	(.38)	(.15)	(.057)	(.47)		

k	N_0	M	λ									D2	D1
			0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1		
7	33	9	0.205	0.139	0.104	0.078	0.129	0.375	0.618	0.784	0.038	6	9
			(.36)	(.33)	(.31)	(.28)	(.43)	(.65)	(.58)	(.43)	(.21)		
	40	10	0.215	0.140	0.097	0.068	0.130	0.411	0.684	0.840	0.032	6	10
			(.47)	(.44)	(.41)	(.37)	(.52)	(.61)	(.42)	(.26)	(.27)		
	47	11	0.223	0.135	0.090	0.065	0.148	0.495	0.763	0.900	0.033	6	11
			(.57)	(.52)	(.49)	(.45)	(.57)	(.52)	(.28)	(.14)	(.33)		
	54	12	0.448	0.295	0.190	0.120	0.185	0.552	0.824	0.939	0.036	7	11
			(.67)	(.63)	(.58)	(.54)	(.7)	(.63)	(.36)	(.18)	(.4)		
	61	13	0.450	0.270	0.160	0.099	0.183	0.586	0.855	0.957	0.028	7	12
			(.74)	(.7)	(.65)	(.6)	(.72)	(.52)	(.24)	(.096)	(.44)		
	68	14	0.656	0.453	0.290	0.183	0.282	0.744	0.943	0.989	0.045	7	12
			(.79)	(.75)	(.7)	(.66)	(.71)	(.41)	(.15)	(.05)	(.48)		
	75	15	0.648	0.412	0.239	0.133	0.210	0.669	0.920	0.982	0.023	8	13
			(.85)	(.81)	(.76)	(.72)	(.8)	(.5)	(.19)	(.07)	(.53)		

Table 6.1: Powers of Procedure 3 at the 5% significance level

(c) N = 1680

k	N_0	M	λ									D2	D1
			0.199	0.249	0.333	0.5	2	3	4	5	1		
3	18	9	0.288	0.223	0.146	0.077	0.124	0.309	0.499	0.637	0.024	7	9
			(.77)	(.72)	(.63)	(.49)	(.53)	(.62)	(.53)	(.42)	(.3)		
	21	10	0.531	0.419	0.279	0.130	0.154	0.381	0.575	0.711	0.034	8	9
			(.88)	(.83)	(.75)	(.59)	(.65)	(.73)	(.63)	(.5)	(.36)		
	24	11	0.493	0.379	0.248	0.114	0.171	0.431	0.646	0.784	0.028	8	10
			(.89)	(.87)	(.79)	(.64)	(.65)	(.62)	(.46)	(.32)	(.4)		
	27	12	0.681	0.558	0.382	0.172	0.205	0.492	0.709	0.835	0.035	9	10
			(.94)	(.92)	(.86)	(.71)	(.74)	(.71)	(.52)	(.38)	(.44)		
	30	13	0.639	0.508	0.335	0.143	0.207	0.523	0.752	0.875	0.028	9	11
			(.93)	(.92)	(.88)	(.75)	(.72)	(.59)	(.38)	(.23)	(.47)		
	33	14	0.781	0.660	0.466	0.205	0.243	0.582	0.798	0.902	0.036	10	11
			(.96)	(.95)	(.92)	(.8)	(.8)	(.66)	(.44)	(.28)	(.51)		
	36	15	0.876	0.776	0.586	0.283	0.337	0.704	0.886	0.956	0.053	10	11
			(.95)	(.94)	(.92)	(.82)	(.76)	(.55)	(.31)	(.17)	(.53)		

k	N_0	M	λ									D2	D1
			0.2	0.253	0.33	0.504	2	3	4	4.98	1		
5	25	9	0.218	0.179	0.135	0.080	0.153	0.398	0.620	0.773	0.040	6	9
			(.41)	(.39)	(.36)	(.29)	(.42)	(.55)	(.46)	(.34)	(.21)		
	30	10	0.212	0.158	0.105	0.051	0.103	0.316	0.538	0.702	0.021	7	10
			(.56)	(.53)	(.48)	(.4)	(.57)	(.7)	(.58)	(.43)	(.28)		
	35	11	0.446	0.364	0.264	0.134	0.224	0.548	0.777	0.895	0.050	7	10
			(.65)	(.62)	(.58)	(.47)	(.62)	(.61)	(.42)	(.26)	(.34)		
	40	12	0.438	0.337	0.226	0.097	0.164	0.468	0.719	0.859	0.030	8	11
			(.75)	(.72)	(.67)	(.56)	(.73)	(.71)	(.5)	(.33)	(.4)		
	45	13	0.419	0.312	0.199	0.080	0.165	0.504	0.757	0.893	0.021	8	12
			(.81)	(.78)	(.73)	(.62)	(.74)	(.6)	(.36)	(.2)	(.45)		
	50	14	0.628	0.503	0.352	0.148	0.270	0.670	0.887	0.963	0.041	8	12
			(.86)	(.83)	(.78)	(.67)	(.72)	(.49)	(.24)	(.11)	(.48)		
	55	15	0.607	0.467	0.314	0.125	0.281	0.707	0.910	0.973	0.034	8	13
			(.89)	(.86)	(.82)	(.71)	(.7)	(.39)	(.16)	(.065)	(.51)		

k	N_0	M	λ									D2	D1
			0.199	0.245	0.34	0.512	2	2.98	4	4.98	1		
7	33	9	-	-	-	-	-	-	-	-	-	6	-
	40	10	0.218	0.179	0.124	0.071	0.154	0.431	0.679	0.828	0.040	6	10
			(.5)	(.49)	(.45)	(.39)	(.55)	(.61)	(.44)	(.28)	(.31)		
	47	11	0.201	0.154	0.094	0.045	0.099	0.343	0.597	0.768	0.019	7	11
			(.62)	(.6)	(.56)	(.49)	(.69)	(.73)	(.53)	(.36)	(.39)		
	54	12	0.442	0.367	0.241	0.122	0.224	0.576	0.822	0.931	0.052	7	11
			(.7)	(.68)	(.64)	(.57)	(.72)	(.63)	(.38)	(.21)	(.45)		
	61	13	0.432	0.340	0.216	0.097	0.221	0.601	0.850	0.947	0.037	7	12
			(.76)	(.75)	(.7)	(.63)	(.73)	(.52)	(.26)	(.12)	(.5)		
	68	14	0.404	0.307	0.173	0.065	0.158	0.518	0.803	0.922	0.018	8	13
			(.83)	(.81)	(.77)	(.7)	(.81)	(.61)	(.32)	(.16)	(.56)		
	75	15	0.613	0.499	0.316	0.129	0.261	0.683	0.915	0.977	0.037	8	13
			(.87)	(.85)	(.82)	(.74)	(.8)	(.5)	(.21)	(.091)	(.6)		

(d) N = 1680000

k	N_0	M	λ								D2	D1
			0.2	0.333	0.5	2	3	4	5	1		
3	18	9	0.292	0.150	0.076	0.123	0.312	0.497	0.639	0.025	7	9
			(.78)	(.63)	(.49)	(.54)	(.62)	(.53)	(.42)	(.3)		
	21	10	0.533	0.285	0.131	0.160	0.380	0.578	0.713	0.035	8	9
			(.88)	(.75)	(.59)	(.65)	(.73)	(.62)	(.5)	(.37)		
	24	11	0.490	0.250	0.116	0.170	0.432	0.642	0.784	0.028	8	10
			(.89)	(.79)	(.64)	(.66)	(.62)	(.46)	(.32)	(.4)		
	27	12	0.677	0.384	0.174	0.210	0.495	0.708	0.834	0.039	9	10
			(.94)	(.86)	(.72)	(.73)	(.71)	(.53)	(.38)	(.45)		
	30	13	0.640	0.339	0.149	0.206	0.524	0.749	0.874	0.029	9	11
			(.93)	(.88)	(.75)	(.72)	(.6)	(.38)	(.23)	(.48)		
	33	14	0.781	0.467	0.207	0.246	0.581	0.796	0.907	0.035	10	11
			(.96)	(.92)	(.8)	(.79)	(.66)	(.44)	(.28)	(.51)		
	36	15	0.744	0.419	0.176	0.238	0.601	0.821	0.922	0.027	10	12
			(.95)	(.92)	(.82)	(.76)	(.55)	(.32)	(.18)	(.54)		

k	N_0	M	λ								D2	D1
			0.2	0.333	0.5	2	3	4	5	1		
5	25	9	0.229	0.134	0.083	0.158	0.400	0.620	0.771	0.043	6	9
			(.43)	(.36)	(.3)	(.42)	(.54)	(.46)	(.34)	(.21)		
	30	10	0.214	0.109	0.054	0.107	0.318	0.537	0.700	0.021	7	10
			(.57)	(.49)	(.41)	(.57)	(.69)	(.58)	(.43)	(.29)		
	35	11	0.457	0.268	0.142	0.230	0.549	0.778	0.896	0.055	7	10
			(.66)	(.58)	(.48)	(.62)	(.61)	(.42)	(.26)	(.35)		
	40	12	0.448	0.234	0.104	0.166	0.469	0.716	0.856	0.030	8	11
			(.76)	(.68)	(.58)	(.73)	(.71)	(.5)	(.33)	(.41)		
	45	13	0.426	0.200	0.084	0.170	0.500	0.756	0.893	0.024	8	12
			(.82)	(.74)	(.64)	(.74)	(.61)	(.36)	(.2)	(.45)		
	50	14	0.628	0.345	0.150	0.224	0.583	0.822	0.932	0.035	9	12
			(.88)	(.8)	(.7)	(.81)	(.69)	(.43)	(.25)	(.51)		
	55	15	0.604	0.304	0.119	0.211	0.601	0.843	0.948	0.025	9	13
			(.91)	(.84)	(.74)	(.8)	(.57)	(.31)	(.15)	(.55)		

k	N_0	M	λ								D2	D1
			0.2	0.333	0.5	2	3	4	5	1		
7	33	9	-	-	-	-	-	-	-	-	6	-
	40	10	0.222	0.131	0.077	0.162	0.438	0.683	0.827	0.044	6	10
			(.51)	(.46)	(.41)	(.56)	(.6)	(.44)	(.28)	(.32)		
	47	11	0.208	0.104	0.048	0.104	0.343	0.595	0.774	0.021	7	11
			(.64)	(.57)	(.51)	(.69)	(.73)	(.54)	(.35)	(.4)		
	54	12	0.197	0.088	0.043	0.123	0.412	0.678	0.841	0.018	7	12
			(.71)	(.66)	(.59)	(.72)	(.63)	(.39)	(.22)	(.47)		
	61	13	0.435	0.222	0.098	0.168	0.497	0.764	0.900	0.031	8	12
			(.79)	(.74)	(.66)	(.81)	(.72)	(.47)	(.27)	(.53)		
	68	14	0.412	0.189	0.076	0.163	0.523	0.796	0.924	0.021	8	13
			(.84)	(.79)	(.72)	(.81)	(.61)	(.33)	(.16)	(.58)		
	75	15	0.620	0.341	0.150	0.269	0.696	0.910	0.977	0.045	8	13
			(.88)	(.83)	(.76)	(.8)	(.5)	(.23)	(.096)	(.62)		

Table 6.2: Powers of Procedure 3 at the 10% significance level

(a) N = 60

k	N_0	M	λ								D2	D1
			0.222	0.308	0.5	2	3	4	4.67	1		
3	18	9	0.727	0.553	0.281	0.372	0.701	0.873	0.930	0.091	6	7
			(.67)	(.56)	(.37)	(.42)	(.41)	(.27)	(.19)	(.18)		
	21	10	0.754	0.564	0.267	0.395	0.747	0.910	0.958	0.076	6	8
			(.73)	(.63)	(.42)	(.41)	(.3)	(.15)	(.084)	(.18)		
	24	11	0.891	0.722	0.368	0.508	0.864	0.969	0.989	0.096	6	8
			(.75)	(.68)	(.46)	(.39)	(.2)	(.067)	(.031)	(.19)		
	27	12	0.966	0.827	0.425	0.508	0.872	0.975	0.992	0.071	7	8
			(.87)	(.82)	(.6)	(.55)	(.29)	(.098)	(.047)	(.23)		
	30	13	1.000	0.919	0.515	0.614	0.939	0.994	0.999	0.081	7	8
			(.86)	(.83)	(.64)	(.5)	(.18)	(.04)	(.014)	(.23)		
	33	14	1.000	0.972	0.611	0.702	0.973	0.999	1.000	0.090	7	8
			(.83)	(.83)	(.68)	(.45)	(.097)	(.013)	(.0028)	(.22)		
	36	15	1.000	1.000	0.696	0.786	0.991	1.000	1.000	0.092	7	8
			(.81)	(.82)	(.7)	(.37)	(.048)	(.0031)	(.00)	(.21)		

k	N_0	M	λ								D2	D1
			0.286	0.286	0.615	2	2.67	3.5	4.57	1		
5	25	9	0.208	0.205	0.099	0.421	0.740	0.926	0.987	0.055	5	7
			(.16)	(.16)	(.073)	(.24)	(.3)	(.21)	(.087)	(.039)		
	30	10	0.276	0.271	0.125	0.575	0.879	0.982	0.999	0.062	5	7
			(.21)	(.2)	(.085)	(.28)	(.25)	(.1)	(.022)	(.036)		
	35	11	0.325	0.328	0.137	0.713	0.957	0.997	1.000	0.060	5	7
			(.24)	(.24)	(.08)	(.28)	(.15)	(.032)	(.00)	(.027)		
	40	12	0.368	0.364	0.140	0.876	0.996	1.000	1.000	0.054	5	6
			(.26)	(.26)	(.07)	(.24)	(.064)	(.00)	(.00)	(.013)		
	45	13	-	-	-	-	-	-	-	-	5	-
			-	-	-	-	-	-	-	-		
	50	14	-	-	-	-	-	-	-	-	5	-
			-	-	-	-	-	-	-	-		
	55	15	-	-	-	-	-	-	-	-	5	-

k	N_0	M	λ								D2	D1
			0	0.455	0.455	1.67	2.5	3.57	5	1		
6	29	9	-	0.142	0.146	0.290	0.761	0.973	1.000	0.080	4	7
			(-)	(.034)	(.033)	(.057)	(.099)	(.038)	(.	(.0088)		
	35	10	-	0.168	0.167	0.404	0.910	0.999	1.000	0.085	4	7
			(-)	(.036)	(.034)	(.068)	(.067)	(.00)	(.00)	(.004)		
	41	11	-	-	-	-	-	-	-	-	4	-
			-	-	-	-	-	-	-	-		
	47	12	-	-	-	-	-	-	-	-	4	-
			-	-	-	-	-	-	-	-		
	53	13	-	-	-	-	-	-	-	-	4	-
			-	-	-	-	-	-	-	-		
	59	14	-	-	-	-	-	-	-	-	4	-
			-	-	-	-	-	-	-	-		
	65	15	-	-	-	-	-	-	-	-	-	-

(b) N = 420

k	N_0	M	λ									D2	D1
			0.199	0.246	0.333	0.5	2	3	4	5	1		
3	18	9	0.628	0.536	0.393	0.229	0.307	0.590	0.775	0.877	0.095	6	8
			(.68)	(.63)	(.54)	(.41)	(.41)	(.4)	(.28)	(.19)	(.25)		
	21	10	0.781	0.688	0.517	0.289	0.337	0.633	0.818	0.907	0.097	7	8
			(.82)	(.77)	(.69)	(.54)	(.55)	(.52)	(.37)	(.25)	(.33)		
	24	11	0.744	0.645	0.476	0.251	0.340	0.660	0.847	0.930	0.082	7	9
			(.81)	(.79)	(.72)	(.58)	(.53)	(.42)	(.25)	(.14)	(.35)		
	27	12	0.856	0.762	0.589	0.314	0.366	0.702	0.876	0.947	0.084	8	9
			(.89)	(.87)	(.82)	(.67)	(.64)	(.51)	(.31)	(.18)	(.41)		
	30	13	0.825	0.727	0.540	0.277	0.363	0.725	0.895	0.962	0.066	8	10
			(.88)	(.86)	(.82)	(.7)	(.61)	(.4)	(.2)	(.1)	(.43)		
	33	14	0.913	0.833	0.660	0.361	0.460	0.824	0.948	0.984	0.091	8	10
			(.85)	(.85)	(.82)	(.71)	(.57)	(.3)	(.13)	(.055)	(.45)		
	36	15	0.955	0.902	0.756	0.448	0.551	0.883	0.977	0.995	0.117	8	10
			(.83)	(.82)	(.81)	(.71)	(.52)	(.23)	(.079)	(.027)	(.46)		

k	N_0	M	λ									D2	D1
			0.2	0.242	0.33	0.516	2	3	3.92	4.94	1		
5	25	9	0.258	0.228	0.186	0.134	0.271	0.584	0.777	0.895	0.090	5	9
			(.33)	(.31)	(.27)	(.21)	(.27)	(.3)	(.22)	(.13)	(.14)		
	30	10	0.419	0.367	0.278	0.158	0.293	0.630	0.827	0.926	0.078	6	9
			(.51)	(.48)	(.43)	(.34)	(.47)	(.48)	(.33)	(.2)	(.24)		
	35	11	0.444	0.383	0.272	0.150	0.306	0.669	0.862	0.949	0.074	6	10
			(.59)	(.56)	(.5)	(.4)	(.49)	(.39)	(.22)	(.11)	(.27)		
	40	12	0.631	0.557	0.426	0.238	0.433	0.801	0.941	0.985	0.113	6	10
			(.65)	(.62)	(.57)	(.46)	(.49)	(.31)	(.14)	(.054)	(.31)		
	45	13	0.632	0.534	0.376	0.177	0.338	0.740	0.916	0.976	0.058	7	11
			(.77)	(.74)	(.68)	(.56)	(.62)	(.4)	(.19)	(.077)	(.39)		
	50	14	0.774	0.691	0.525	0.258	0.448	0.848	0.963	0.993	0.087	7	11
			(.81)	(.78)	(.73)	(.6)	(.6)	(.3)	(.12)	(.04)	(.41)		
	55	15	0.867	0.801	0.652	0.346	0.560	0.915	0.986	0.998	0.120	7	11
			(.84)	(.81)	(.76)	(.63)	(.56)	(.22)	(.07)	(.02)	(.44)		

k	N_0	M	λ									D2	D1
			0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1		
7	33	9	0.226	0.170	0.137	0.113	0.202	0.517	0.764	0.892	0.070	5	9
			(.32)	(.3)	(.27)	(.24)	(.34)	(.43)	(.31)	(.18)	(.17)		
	40	10	0.256	0.185	0.145	0.122	0.243	0.596	0.829	0.934	0.083	5	10
			(.42)	(.39)	(.35)	(.32)	(.4)	(.37)	(.2)	(.096)	(.23)		
	47	11	0.444	0.323	0.231	0.165	0.259	0.642	0.873	0.959	0.071	6	10
			(.57)	(.52)	(.49)	(.45)	(.57)	(.52)	(.28)	(.13)	(.33)		
	54	12	0.462	0.309	0.208	0.145	0.261	0.672	0.900	0.971	0.060	6	11
			(.64)	(.6)	(.56)	(.51)	(.6)	(.42)	(.18)	(.071)	(.37)		
	61	13	0.649	0.487	0.352	0.249	0.382	0.811	0.964	0.993	0.096	6	11
			(.7)	(.65)	(.61)	(.56)	(.59)	(.32)	(.11)	(.035)	(.41)		
	68	14	0.654	0.453	0.290	0.181	0.286	0.741	0.944	0.990	0.049	7	12
			(.79)	(.75)	(.7)	(.65)	(.71)	(.42)	(.14)	(.047)	(.48)		
	75	15	0.788	0.608	0.431	0.284	0.397	0.852	0.981	0.998	0.072	7	12
			(.83)	(.79)	(.75)	(.7)	(.68)	(.32)	(.087)	(.025)	(.51)		

Table 6.2: Powers of Procedure 3 at the 10% significance level

(c) N = 1680

k	N_0	M	λ									D2	D1
			0.199	0.249	0.333	0.5	2	3	4	5	1		
3	18	9	0.597	0.492	0.352	0.186	0.218	0.455	0.645	0.771	0.063	7	8
			(.78)	(.72)	(.64)	(.49)	(.54)	(.62)	(.54)	(.43)	(.3)		
	21	10	0.776	0.679	0.525	0.298	0.345	0.639	0.814	0.905	0.103	7	8
			(.81)	(.77)	(.69)	(.54)	(.54)	(.52)	(.37)	(.25)	(.34)		
	24	11	0.745	0.641	0.479	0.264	0.344	0.669	0.848	0.927	0.090	7	9
			(.81)	(.78)	(.71)	(.58)	(.53)	(.42)	(.25)	(.15)	(.36)		
	27	12	0.853	0.760	0.590	0.326	0.373	0.704	0.875	0.947	0.096	8	9
			(.89)	(.87)	(.81)	(.68)	(.64)	(.51)	(.32)	(.19)	(.42)		
	30	13	0.824	0.721	0.546	0.284	0.373	0.721	0.893	0.960	0.077	8	10
			(.87)	(.85)	(.82)	(.69)	(.61)	(.41)	(.21)	(.11)	(.45)		
	33	14	0.900	0.816	0.646	0.352	0.401	0.758	0.911	0.969	0.080	9	10
			(.92)	(.91)	(.88)	(.76)	(.7)	(.48)	(.26)	(.14)	(.49)		
	36	15	0.948	0.887	0.740	0.429	0.494	0.839	0.955	0.988	0.107	9	10
			(.9)	(.89)	(.87)	(.78)	(.66)	(.38)	(.18)	(.078)	(.51)		

k	N_0	M	λ									D2	D1
			0.2	0.253	0.33	0.504	2	3	4	4.98	1		
5	25	9	0.221	0.178	0.135	0.080	0.155	0.396	0.623	0.767	0.040	6	9
			(.42)	(.39)	(.36)	(.29)	(.42)	(.54)	(.46)	(.34)	(.21)		
	30	10	0.438	0.371	0.294	0.181	0.306	0.631	0.836	0.926	0.091	6	9
			(.52)	(.49)	(.45)	(.37)	(.48)	(.48)	(.33)	(.2)	(.26)		
	35	11	0.467	0.381	0.293	0.172	0.316	0.667	0.868	0.947	0.083	6	10
			(.61)	(.57)	(.52)	(.43)	(.49)	(.4)	(.22)	(.11)	(.3)		
	40	12	0.638	0.544	0.423	0.237	0.360	0.727	0.905	0.969	0.094	7	10
			(.73)	(.69)	(.64)	(.54)	(.63)	(.51)	(.29)	(.15)	(.39)		
	45	13	0.644	0.538	0.398	0.200	0.350	0.738	0.917	0.975	0.075	7	11
			(.78)	(.75)	(.7)	(.59)	(.62)	(.41)	(.19)	(.085)	(.42)		
	50	14	0.784	0.685	0.545	0.300	0.463	0.842	0.967	0.993	0.114	7	11
			(.82)	(.79)	(.74)	(.64)	(.6)	(.32)	(.12)	(.045)	(.45)		
	55	15	0.777	0.659	0.497	0.238	0.379	0.790	0.949	0.989	0.064	8	12
			(.88)	(.86)	(.81)	(.71)	(.7)	(.4)	(.16)	(.065)	(.52)		

k	N_0	M	λ									D2	D1
			0.199	0.245	0.34	0.512	2	2.98	4	4.98	1		
7	33	9	0.231	0.205	0.159	0.118	0.235	0.532	0.760	0.880	0.082	5	9
			(.34)	(.33)	(.3)	(.26)	(.37)	(.43)	(.32)	(.2)	(.2)		
	40	10	0.222	0.178	0.123	0.071	0.157	0.429	0.676	0.829	0.039	6	10
			(.5)	(.48)	(.45)	(.39)	(.55)	(.61)	(.44)	(.28)	(.31)		
	47	11	0.450	0.392	0.283	0.174	0.308	0.661	0.873	0.953	0.092	6	10
			(.59)	(.58)	(.53)	(.47)	(.59)	(.52)	(.3)	(.16)	(.37)		
	54	12	0.459	0.381	0.263	0.151	0.300	0.685	0.893	0.966	0.076	6	11
			(.66)	(.64)	(.6)	(.53)	(.61)	(.42)	(.2)	(.089)	(.42)		
	61	13	0.637	0.555	0.408	0.227	0.369	0.759	0.934	0.982	0.098	7	11
			(.77)	(.74)	(.71)	(.63)	(.72)	(.52)	(.26)	(.12)	(.5)		
	68	14	0.636	0.543	0.369	0.185	0.343	0.755	0.940	0.986	0.071	7	12
			(.81)	(.79)	(.75)	(.68)	(.71)	(.42)	(.17)	(.067)	(.55)		
	75	15	0.774	0.690	0.513	0.281	0.462	0.862	0.975	0.996	0.110	7	12
			(.84)	(.83)	(.79)	(.72)	(.68)	(.33)	(.11)	(.038)	(.58)		

(d) N = 1680000

k	N_0	M	λ								D2	D1
			0.2	0.333	0.5	2	3	4	5	1		
3	18	9	0.595	0.350	0.185	0.218	0.457	0.644	0.773	0.063	7	8
			(.77)	(.64)	(.49)	(.53)	(.61)	(.53)	(.42)	(.3)		
	21	10	0.779	0.527	0.299	0.349	0.641	0.815	0.904	0.108	7	8
			(.81)	(.69)	(.54)	(.54)	(.52)	(.38)	(.26)	(.34)		
	24	11	0.745	0.482	0.267	0.346	0.668	0.844	0.925	0.089	7	9
			(.81)	(.71)	(.57)	(.53)	(.42)	(.25)	(.15)	(.37)		
	27	12	0.851	0.592	0.330	0.381	0.707	0.874	0.946	0.096	8	9
			(.88)	(.81)	(.67)	(.64)	(.51)	(.32)	(.19)	(.43)		
	30	13	0.824	0.546	0.289	0.372	0.722	0.892	0.959	0.076	8	10
			(.87)	(.82)	(.69)	(.61)	(.41)	(.21)	(.11)	(.45)		
	33	14	0.905	0.663	0.378	0.473	0.813	0.944	0.984	0.108	8	10
			(.84)	(.81)	(.7)	(.56)	(.32)	(.14)	(.061)	(.47)		
	36	15	0.878	0.598	0.306	0.395	0.763	0.921	0.975	0.066	9	11
			(.9)	(.87)	(.78)	(.65)	(.38)	(.18)	(.08)	(.52)		

k	N_0	M	λ								D2	D1
			0.2	0.333	0.5	2	3	4	5	1		
5	25	9	0.275	0.204	0.149	0.283	0.577	0.783	0.892	0.101	5	9
			(.34)	(.29)	(.23)	(.29)	(.31)	(.22)	(.14)	(.16)		
	30	10	0.441	0.297	0.187	0.305	0.636	0.834	0.928	0.097	6	9
			(.52)	(.45)	(.38)	(.48)	(.48)	(.33)	(.2)	(.27)		
	35	11	0.471	0.295	0.176	0.317	0.669	0.864	0.947	0.087	6	10
			(.61)	(.53)	(.44)	(.49)	(.4)	(.23)	(.12)	(.31)		
	40	12	0.644	0.423	0.243	0.367	0.728	0.905	0.968	0.098	7	10
			(.74)	(.64)	(.55)	(.63)	(.52)	(.29)	(.16)	(.39)		
	45	13	0.650	0.402	0.211	0.358	0.740	0.917	0.974	0.079	7	11
			(.78)	(.7)	(.6)	(.62)	(.41)	(.19)	(.086)	(.43)		
	50	14	0.785	0.544	0.310	0.471	0.845	0.964	0.993	0.122	7	11
			(.81)	(.74)	(.64)	(.6)	(.32)	(.12)	(.048)	(.47)		
	55	15	0.786	0.515	0.276	0.465	0.855	0.970	0.994	0.103	7	12
			(.83)	(.77)	(.67)	(.57)	(.25)	(.08)	(.027)	(.49)		

k	N_0	M	λ								D2	D1
			0.2	0.333	0.5	2	3	4	5	1		
7	33	9	0.238	0.174	0.126	0.237	0.533	0.758	0.884	0.087	5	9
			(.35)	(.31)	(.27)	(.37)	(.43)	(.32)	(.2)	(.21)		
	40	10	0.228	0.132	0.077	0.159	0.437	0.679	0.829	0.042	6	10
			(.51)	(.46)	(.41)	(.56)	(.61)	(.44)	(.28)	(.32)		
	47	11	0.457	0.302	0.189	0.314	0.669	0.873	0.953	0.099	6	10
			(.6)	(.54)	(.49)	(.6)	(.52)	(.31)	(.16)	(.38)		
	54	12	0.463	0.281	0.164	0.313	0.688	0.892	0.966	0.082	6	11
			(.67)	(.62)	(.55)	(.61)	(.43)	(.2)	(.088)	(.43)		
	61	13	0.440	0.232	0.113	0.225	0.603	0.845	0.948	0.045	7	12
			(.77)	(.71)	(.65)	(.73)	(.52)	(.27)	(.13)	(.52)		
	68	14	0.639	0.394	0.201	0.353	0.758	0.935	0.986	0.077	7	12
			(.82)	(.76)	(.69)	(.71)	(.42)	(.18)	(.071)	(.57)		
	75	15	0.630	0.355	0.172	0.351	0.780	0.949	0.990	0.065	7	13
			(.85)	(.8)	(.73)	(.68)	(.33)	(.11)	(.038)	(.6)		

Table 6.3: Expected sample size when using Procedure 3 at the 5% significance level

(a) N = 60

k	N_0	M	λ								
			0.222	0.222	0.308	0.5	2	3	4	4.67	1
3	18	9	16.23	16.23	16.60	17.17	16.38	14.41	12.96	12.24	17.69
	21	10	18.39	18.40	18.93	19.78	18.58	15.74	13.80	12.88	20.59
	24	11	20.45	20.44	21.12	22.34	20.53	16.77	14.30	13.23	23.45
	27	12	22.34	22.38	23.24	24.79	22.18	17.45	14.57	13.37	26.37
	30	13	24.78	24.79	25.75	27.62	24.70	19.57	16.53	15.19	29.46
	33	14	26.80	26.77	27.95	30.10	26.27	20.15	16.70	15.28	32.40
	36	15	28.75	28.72	29.92	32.53	27.63	20.48	16.78	15.29	35.36

k	N_0	M	λ								
			0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1
5	25	9	24.21	24.22	24.23	24.54	22.65	19.79	16.52	13.66	24.71
	30	10	28.74	28.71	28.71	29.27	25.78	21.31	17.06	13.81	29.55
	35	11	33.11	33.13	33.15	33.93	28.31	22.15	17.29	13.84	34.40
	40	12	37.42	37.37	37.43	38.59	29.87	22.51	17.30	13.84	39.18
	45	13	-	-	-	-	-	-	-	-	-
	50	14	-	-	-	-	-	-	-	-	-
	55	15	-	-	-	-	-	-	-	-	-

k	N_0	M	λ								
			0	0.455	0.455	0.455	1.67	2.5	3.57	5	1
6	29	9	NaN	28.63	28.63	28.63	27.91	23.91	18.70	14.49	28.85
	35	10	NaN	34.48	34.45	34.46	32.92	26.02	19.19	14.53	34.76
	41	11	-	-	-	-	-	-	-	-	-
	47	12	-	-	-	-	-	-	-	-	-
	53	13	-	-	-	-	-	-	-	-	-
	59	14	-	-	-	-	-	-	-	-	-
	65	15	-	-	-	-	-	-	-	-	-

(b) N = 420

k	N_0	M	λ								
			0.199	0.246	0.333	0.5	2	3	4	5	1
3	18	9	15.80	16.03	16.40	16.89	16.21	14.37	12.97	11.97	17.46
	21	10	18.02	18.29	18.77	19.47	18.59	16.29	14.63	13.52	20.28
	24	11	20.00	20.39	21.01	21.97	20.69	17.68	15.62	14.23	23.06
	27	12	22.11	22.56	23.31	24.49	22.94	19.52	17.30	15.82	25.86
	30	13	24.11	24.56	25.49	26.89	24.96	20.76	18.09	16.32	28.65
	33	14	26.04	26.60	27.58	29.22	26.85	21.83	18.67	16.66	31.39
	36	15	28.16	28.79	29.86	31.70	29.11	23.76	20.53	18.35	34.23

k	N_0	M	λ								
			0.2	0.242	0.33	0.516	2	3	3.92	4.94	1
5	25	9	23.62	23.71	23.86	24.11	22.84	19.64	17.04	15.05	24.39
	30	10	27.76	27.84	28.13	28.48	26.35	21.59	18.27	15.72	29.04
	35	11	31.94	32.16	32.45	33.03	30.20	24.54	20.74	17.94	33.78
	40	12	35.84	36.09	36.50	37.31	33.39	26.15	21.62	18.44	38.36
	45	13	39.84	40.16	40.71	41.71	37.02	29.00	24.13	20.69	43.05
	50	14	43.66	44.02	44.73	45.98	39.96	30.48	24.85	21.05	47.59
	55	15	47.40	47.86	48.62	50.16	42.72	31.63	25.34	21.17	52.17

k	N_0	M	λ								
			0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1
7	33	9	31.45	31.59	31.67	31.82	30.40	25.86	21.58	18.60	32.16
	40	10	37.36	37.59	37.81	38.02	35.58	28.81	23.15	19.57	38.65
	47	11	43.09	43.45	43.75	44.03	40.26	31.26	24.40	20.18	44.99
	54	12	49.05	49.48	49.89	50.27	45.50	34.98	27.54	22.98	51.56
	61	13	54.57	55.19	55.71	56.18	49.87	36.95	28.39	23.35	57.87
	68	14	60.07	60.79	61.40	62.06	53.95	38.66	29.01	23.63	64.18
	75	15	65.82	66.67	67.46	68.13	58.95	42.50	32.38	26.44	70.71

(c) N = 1680

k	N_0	M	λ								
			0.199	0.249	0.333	0.5	2	3	4	5	1
3	18	9	15.76	16.02	16.35	16.86	16.17	14.38	12.98	11.99	17.43
	21	10	17.94	18.26	18.72	19.44	18.53	16.27	14.66	13.52	20.23
	24	11	19.94	20.34	20.94	21.90	20.65	17.66	15.60	14.23	23.00
	27	12	22.05	22.49	23.24	24.39	22.93	19.54	17.33	15.80	25.79
	30	13	24.01	24.56	25.39	26.80	24.89	20.82	18.09	16.34	28.54
	33	14	26.14	26.65	27.65	29.25	27.15	22.69	19.85	18.01	31.34
	36	15	28.09	28.74	29.77	31.61	29.07	23.86	20.51	18.38	34.09

k	N_0	M	λ								
			0.2	0.253	0.33	0.504	2	3	4	4.98	1
5	25	9	23.54	23.63	23.76	24.01	22.76	19.65	16.94	15.00	24.34
	30	10	27.74	27.95	28.11	28.52	26.65	22.53	19.29	17.08	29.04
	35	11	31.74	31.96	32.25	32.80	30.11	24.54	20.55	17.87	33.62
	40	12	35.70	36.03	36.38	37.16	33.68	27.25	22.86	19.99	38.20
	45	13	39.52	39.85	40.38	41.32	36.91	29.03	23.94	20.61	42.75
	50	14	43.26	43.69	44.25	45.44	39.83	30.47	24.59	21.02	47.21
	55	15	46.89	47.38	48.06	49.52	42.59	31.68	25.09	21.13	51.64

k	N_0	M	λ								
			0.199	0.245	0.34	0.512	2	2.98	4	4.98	1
7	33	9	-	-	-	-	-	-	-	-	-
	40	10	37.12	37.22	37.45	37.83	34.92	28.58	23.27	19.89	38.36
	47	11	42.90	43.09	43.41	43.94	40.00	32.06	26.33	22.52	44.77
	54	12	48.48	48.70	49.13	49.81	44.39	34.56	27.60	23.33	51.01
	61	13	53.87	54.21	54.74	55.61	48.56	36.59	28.56	23.76	57.09
	68	14	59.43	59.77	60.43	61.54	53.26	40.26	31.81	26.64	63.38
	75	15	64.81	65.18	65.99	67.34	57.31	42.17	32.63	26.99	69.45

(d) N = 1680000

k	N_0	M	λ								
			0.2	0.25	0.333	0.5	2	3	4	5	1
3	18	9	15.77	16.00	16.34	16.86	16.19	14.37	12.98	11.96	17.41
	21	10	17.91	18.26	18.73	19.43	18.52	16.26	14.63	13.54	20.22
	24	11	19.94	20.36	20.96	21.88	20.64	17.69	15.64	14.24	22.99
	27	12	22.03	22.49	23.19	24.39	22.92	19.56	17.35	15.84	25.76
	30	13	24.01	24.52	25.37	26.76	24.91	20.82	18.13	16.36	28.50
	33	14	26.13	26.67	27.61	29.24	27.13	22.69	19.87	18.03	31.30
	36	15	28.07	28.73	29.73	31.55	29.10	23.84	20.53	18.37	34.04

k	N_0	M	λ								
			0.2	0.25	0.333	0.5	2	3	4	5	1
5	25	9	23.52	23.61	23.75	23.97	22.75	19.62	16.95	14.98	24.31
	30	10	27.70	27.86	28.05	28.45	26.63	22.54	19.25	17.06	29.01
	35	11	31.64	31.84	32.21	32.75	30.07	24.57	20.52	17.89	33.55
	40	12	35.61	35.92	36.33	37.06	33.62	27.23	22.88	20.05	38.16
	45	13	39.40	39.73	40.31	41.20	36.90	29.03	23.89	20.62	42.62
	50	14	43.24	43.60	44.25	45.39	40.31	31.74	26.32	22.81	47.17
	55	15	46.91	47.40	48.10	49.44	43.38	33.41	27.24	23.30	51.60

k	N_0	M	λ								
			0.2	0.25	0.333	0.5	2	3	4	5	1
7	33	9	-	-	-	-	-	-	-	-	-
	40	10	37.00	37.13	37.31	37.69	34.86	28.42	23.29	19.86	38.29
	47	11	42.75	42.95	43.27	43.74	39.99	32.01	26.28	22.50	44.59
	54	12	48.24	48.51	48.87	49.57	44.38	34.47	27.70	23.29	50.79
	61	13	53.70	54.08	54.57	55.46	49.05	37.97	30.68	26.05	56.98
	68	14	59.14	59.45	60.06	61.14	53.24	40.15	31.96	26.64	63.11
	75	15	64.34	64.84	65.48	66.77	57.16	42.13	32.70	26.98	69.07

Table 6.4: Expected sample size when using Procedure 3 at the 10% significance level

(a) N = 60

k	N_0	M	λ								
			0.222	0.222	0.308	0.5	2	3	4	4.67	1
3	18	9	15.97	15.98	16.34	17.00	16.04	13.73	12.00	11.19	17.59
	21	10	17.92	17.90	18.49	19.44	17.93	14.63	12.42	11.49	20.40
	24	11	19.67	19.68	20.43	21.74	19.55	15.24	12.67	11.52	23.18
	27	12	22.38	22.40	23.20	24.78	22.22	17.46	14.63	13.38	26.38
	30	13	24.18	24.21	25.23	27.13	23.69	17.92	14.77	13.45	29.23
	33	14	26.01	25.99	27.15	29.47	24.95	18.20	14.75	13.48	32.11
	36	15	27.73	27.70	28.99	31.65	25.87	18.33	14.82	13.44	34.98

k	N_0	M	λ								
			0.286	0.286	0.286	0.615	2	2.67	3.5	4.57	1
5	25	9	24.25	24.21	24.24	24.55	22.64	19.84	16.56	13.64	24.71
	30	10	28.73	28.73	28.70	29.28	25.80	21.31	17.03	13.84	29.54
	35	11	33.11	33.11	33.11	33.91	28.28	22.17	17.26	13.83	34.37
	40	12	37.43	37.36	37.43	38.56	29.95	22.53	17.33	13.82	39.18
	45	13	-	-	-	-	-	-	-	-	-
	50	14	-	-	-	-	-	-	-	-	-
	55	15	-	-	-	-	-	-	-	-	-

k	N_0	M	λ								
			0	0.455	0.455	0.455	1.67	2.5	3.57	5	1
6	29	9	NaN	27.65	27.65	27.66	26.56	21.53	15.85	11.97	28.13
	35	10	NaN	32.87	32.88	32.88	30.65	22.65	16.00	12.00	33.63
	41	11	-	-	-	-	-	-	-	-	-
	47	12	-	-	-	-	-	-	-	-	-
	53	13	-	-	-	-	-	-	-	-	-
	59	14	-	-	-	-	-	-	-	-	-
	65	15	-	-	-	-	-	-	-	-	-

(b) N = 420

k	N_0	M	λ								
			0.199	0.246	0.333	0.5	2	3	4	5	1
3	18	9	15.47	15.71	16.11	16.66	15.87	13.69	12.05	10.93	17.31
	21	10	17.81	18.11	18.61	19.37	18.33	15.74	13.90	12.60	20.21
	24	11	19.68	20.06	20.70	21.73	20.26	16.81	14.50	12.99	22.91
	27	12	21.92	22.39	23.15	24.34	22.68	18.84	16.29	14.65	25.79
	30	13	23.79	24.30	25.18	26.66	24.47	19.76	16.80	14.93	28.51
	33	14	25.56	26.16	27.15	28.86	26.10	20.54	17.13	15.06	31.23
	36	15	27.29	27.92	29.01	30.98	27.56	21.04	17.31	15.13	33.85

k	N_0	M	λ								
			0.2	0.242	0.33	0.516	2	3	3.92	4.94	1
5	25	9	23.15	23.30	23.42	23.72	22.17	18.46	15.58	13.36	24.08
	30	10	27.75	27.83	28.06	28.52	26.33	21.63	18.24	15.73	29.03
	35	11	31.62	31.79	32.12	32.74	29.54	23.18	19.02	16.11	33.54
	40	12	35.28	35.54	35.94	36.80	32.36	24.22	19.42	16.24	37.95
	45	13	39.66	39.94	40.47	41.44	36.31	27.43	22.23	18.69	42.87
	50	14	43.32	43.68	44.35	45.59	38.91	28.40	22.66	18.75	47.33
	55	15	46.86	47.28	48.15	49.61	41.23	29.09	22.82	18.88	51.71

k	N_0	M	λ								
			0.176	0.269	0.364	0.462	1.92	2.94	4	5.05	1
7	33	9	31.15	31.29	31.40	31.52	29.94	24.78	20.07	16.80	31.91
	40	10	36.80	37.03	37.27	37.47	34.65	27.09	20.91	17.23	38.14
	47	11	43.12	43.45	43.76	44.03	40.36	31.17	24.34	20.15	44.96
	54	12	48.62	49.08	49.51	49.86	44.69	32.96	25.11	20.40	51.23
	61	13	54.07	54.51	55.10	55.63	48.69	34.40	25.47	20.62	57.34
	68	14	60.21	60.78	61.39	62.05	54.06	38.58	29.01	23.61	64.15
	75	15	65.51	66.28	67.04	67.87	57.85	39.78	29.28	23.64	70.43

(c) N = 1680

k	N_0	M	λ								
			0.199	0.249	0.333	0.5	2	3	4	5	1
3	18	9	15.76	16.01	16.36	16.85	16.20	14.39	12.98	12.00	17.43
	21	10	17.75	18.08	18.56	19.33	18.32	15.75	13.89	12.57	20.18
	24	11	19.58	20.03	20.65	21.63	20.24	16.83	14.50	12.98	22.85
	27	12	21.86	22.33	23.08	24.22	22.64	18.85	16.35	14.68	25.72
	30	13	23.69	24.24	25.07	26.53	24.45	19.80	16.87	14.95	28.43
	33	14	25.90	26.51	27.48	29.10	26.79	21.81	18.66	16.69	31.27
	36	15	27.79	28.41	29.48	31.39	28.55	22.69	19.05	16.87	33.97

k	N_0	M	λ								
			0.2	0.253	0.33	0.504	2	3	4	4.98	1
5	25	9	23.53	23.64	23.76	23.99	22.77	19.65	16.89	14.96	24.33
	30	10	27.56	27.72	27.91	28.32	26.29	21.66	18.06	15.67	28.92
	35	11	31.38	31.58	31.88	32.48	29.39	23.18	18.80	15.97	33.35
	40	12	35.52	35.84	36.22	36.99	33.32	26.20	21.32	18.36	38.12
	45	13	39.23	39.63	40.11	41.09	36.17	27.46	22.03	18.64	42.53
	50	14	42.80	43.25	43.85	45.13	38.84	28.55	22.37	18.74	46.87
	55	15	46.89	47.36	48.09	49.51	42.65	31.66	25.09	21.21	51.65

k	N_0	M	λ								
			0.199	0.245	0.34	0.512	2	2.98	4	4.98	1
7	33	9	30.96	31.03	31.17	31.40	29.43	24.58	20.07	17.09	31.74
	40	10	37.11	37.28	37.44	37.81	34.95	28.52	23.29	19.87	38.40
	47	11	42.68	42.90	43.19	43.73	39.39	30.81	24.45	20.44	44.59
	54	12	48.05	48.26	48.75	49.37	43.57	32.66	25.21	20.85	50.66
	61	13	53.85	54.15	54.73	55.61	48.60	36.64	28.59	23.80	57.13
	68	14	59.12	59.54	60.19	61.29	52.45	38.39	29.31	24.09	63.20
	75	15	64.39	64.75	65.51	66.86	56.03	39.54	29.66	24.21	69.15

(d) N = 1680000

k	N_0	M	λ								
			0.2	0.25	0.333	0.5	2	3	4	5	1
3	18	9	15.78	16.00	16.33	16.84	16.18	14.37	12.98	11.98	17.42
	21	10	17.75	18.07	18.54	19.29	18.29	15.75	13.90	12.62	20.15
	24	11	19.58	19.96	20.60	21.63	20.21	16.83	14.53	12.99	22.82
	27	12	21.82	22.31	23.06	24.23	22.60	18.85	16.35	14.69	25.68
	30	13	23.71	24.18	25.03	26.51	24.44	19.77	16.81	14.94	28.39
	33	14	25.43	26.03	27.04	28.74	26.05	20.53	17.14	15.12	31.03
	36	15	27.76	28.40	29.44	31.35	28.54	22.72	19.09	16.85	33.92

k	N_0	M	λ								
			0.2	0.25	0.333	0.5	2	3	4	5	1
5	25	9	23.03	23.14	23.30	23.55	22.09	18.46	15.40	13.31	23.97
	30	10	27.50	27.68	27.88	28.26	26.27	21.65	18.09	15.62	28.88
	35	11	31.29	31.50	31.84	32.38	29.38	23.22	18.78	16.02	33.27
	40	12	35.42	35.72	36.17	36.89	33.30	26.21	21.42	18.37	38.02
	45	13	39.13	39.48	40.02	40.98	36.19	27.54	21.96	18.62	42.40
	50	14	42.69	43.02	43.68	44.87	38.83	28.66	22.36	18.79	46.77
	55	15	46.00	46.61	47.34	48.72	41.17	29.39	22.67	18.91	51.09

k	N_0	M	λ								
			0.2	0.25	0.333	0.5	2	3	4	5	1
7	33	9	30.88	30.97	31.11	31.32	29.37	24.43	20.10	17.02	31.71
	40	10	37.03	37.16	37.36	37.69	34.90	28.46	23.33	19.88	38.28
	47	11	42.53	42.72	42.95	43.53	39.41	30.75	24.46	20.47	44.48
	54	12	47.82	47.96	48.43	49.26	43.49	32.56	25.29	20.87	50.39
	61	13	53.60	53.89	54.44	55.29	48.49	36.50	28.66	23.78	56.81
	68	14	58.76	59.14	59.75	60.89	52.36	38.13	29.33	24.02	62.80
	75	15	63.95	64.35	65.14	66.28	55.99	39.60	29.78	24.20	68.68

7 Directions for Future Research

Inverse sampling procedures are successfully used in homogeneity testing to test whether some parameters in a multivariate hypergeometric distribution are equal or not. The effectiveness of the procedures can be checked by the power of procedures under the alternative hypothesis. The alternative hypothesis we used in our research is still simple. We need more complex situations to check the powers of the procedures.

Although the 3 procedures are successfully used in homogeneity testing, we still could improve the procedures and find even more powerful procedures.

The target and transfer probabilities may be used in other complex problems, so we can continue this kind of research for new complex problems.

The fast algorithms developed can be used in many fields of statistics like order statistics and nonparametric problems.

Continue new theoretical researches on the effective methods, for example the approximation in Procedure 2.

Homogeneity testing in other multivariate distributions can also be studied.

APPENDIX

A. DEFINITIONS, THEOREM PROOFS AND EXAMPLES

A.1 Proofs and Examples from Chapter 3

A.1.1 Proof of theorem 3.3

The probability of $(X_{1t}, X_{2t}, ..., X_{kt})$ being a potential target point $(x_1, x_2, ..., x_k)$ with $X_{(k)t} = x_i$ and $X_{(k-1)t} = x_j$ is given by

$$p(x_1, x_2, ..., x_k) = \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}...\binom{M_k}{x_k}}{\binom{N}{x_1+...+x_{k-1}+x_k}},$$

where $x_i = x_j + D_2 - 1$.

The probability of $(X_{1t}, X_{2t}, ..., X_{kt})$ transferring from a potential target point to a target point is

$$\frac{M_i - x_i}{N - x_1 - ... - x_{k-1} - x_k}.$$

So the target probability at $X_{1t} = x_1, X_{2t} = x_2, ..., X_{kt} = x_k$ is given by

$$B_i(x_1, x_2, ..., x_k) = \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}...\binom{M_k}{x_k}}{\binom{N}{x_1+...+x_{k-1}+x_k}} \frac{M_i - x_i}{N - x_1 - ... - x_{k-1} - x_k}.$$

A.1.2 Proof of theorem 3.5

Suppose $X_{kt} = x_k = X_{(k)t}$ is the largest value of a sample $(X_{1t}, X_{2t}, ..., X_{kt}) = (x_1, x_2, ..., x_k)$, and $X_{k-1,\ t} = x_{k-1} = X_{(k-1)t}$ is the second largest value of that sample. The sample is in layer N_1, so $x_1 + x_2 + ... + x_k = N_1$. Since the condition of Procedure 2 is $X_{(k)t} - X_{(k-1)t} \geq D_2$, and $D_2 > 1$, so when $(x_1, x_2, ..., x_k)$ is a potential target point, there is only one largest value in $x_1, x_2, ..., x_k$, i.e., $x_k > x_{k-1}$. And this largest value is x_k. There can be $j = 1, 2, ..., k-1$ second largest variables tied with x_{k-1}. Suppose that $x_{k-1}, x_{k-2}, ..., x_{k-j}$ are such that $x_{k-1} =$

$x_{k-2} = ... = x_{k-j},\ j = 1, 2, ..., k-1$. Because these j variables with identical values can be any j from the $k-1$ variables, the number of ways this can occur is enumerated by the following sum,

$$\sum_{\substack{j=1}}^{k-1} \binom{k-1}{j}$$
$$x_k = x_{k-1} + D_2 - 1$$
$$x_{k-1} = x_{k-2} = ... = x_{k-j}$$
$$x_1 + x_2 + ... + x_k = N_1$$

The remaining $(k-1-j)$ variables are less than x_{k-1}, i.e., $x_i < x_{k-1}, i = 1, 2, ..., k-j-1$. All of these possibilities are accounted for in the following sum,

$$\sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} \cdots \sum_{x_1=0}^{x_{k-1}-1}$$

Since $x_k = x_{k-1} + D_2 - 1$ and $x_1 + x_2 + ... + x_k = N_1$, we have that $x_1 + x_2 + ... + 2x_{k-1} + D_2 - 1 = N_1$. And $x_1 + x_2 + ... + x_{k-2} \geq 0$, so $2x_{k-1} + D_2 - 1 \leq N_1$, and hence $0 \leq x_{k-1} \leq \frac{1}{2}(N_1 - D_2 + 1)$. Therefore all of the potential target points of $X_{1t}, X_{2t}, ..., X_{kt}$ in layer N_1 with $X_{(k)} = x_k$ can be enumerated with the following sum

$$\sum_{\substack{x_{k-1}=0 \\ x_k = x_{k-1} + D_2 - 1 \\ x_{k-1} = x_{k-2} = ... = x_{k-j} \\ x_1 + x_2 + ... + x_k = N_1}}^{\frac{1}{2}(N_1 - D_2 + 1)} \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} \cdots \sum_{x_1=0}^{x_{k-1}-1}$$

And since each of $X_{1t}, X_{2t}, ..., X_{kt}$ have the same probability of being $X_{(k)}$, the target probability in layer N_1 is given by

$$C_{N_1} = k\{\sum_{x_{k-1}=0}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} \cdots \sum_{x_1=0}^{x_{k-1}-1} B_k(x_1, ..., x_{k-1}, x_k)\}.$$

$$\begin{array}{l} x_k = x_{k-1} + D_2 - 1 \\ x_{k-1} = x_{k-2} = ... = x_{k-j} \\ x_1 + x_2 + ... + x_k = N_1 \end{array}$$

A.1.3 Proof of Theorem 3.7

From theorem 3.5, we know that the target probability in layer N_1 is given by

$$C_{N_1} = k\{\sum_{x_{k-1}=0}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} \cdots \sum_{x_1=0}^{x_{k-1}-1} B_k(x_1, ..., x_{k-1}, x_k)\},$$

$$\begin{array}{l} x_k = x_{k-1} + D_2 - 1 \\ x_{k-1} = x_{k-2} = ... = x_{k-j} \\ x_1 + x_2 + ... + x_k = N_1 \end{array}$$

where $B_k(x_1, ..., x_{k-1}, x_k)$ is the target probability at $X_{1t} = x_1, X_{2t} = x_2, ..., X_{kt} = x_{k-1} + D_2 - 1$.

In this formula, every point $(x_1, x_2, ..., x_k)$ is a potential target point since $x_k = x_{k-1} + D_2 - 1$. If $(x_1, x_2, ..., x_k)$ becomes a target point, then the new value of x_k is $x_k^c = x_k + 1$ since x_k is the largest value at this point. So $(x_1, x_2, ..., x_k^c)$ is a target point. After we get the target point we still continue taking observations one at a time until the next potential target point. The additional values in categories $1, 2, ..., k$ are denoted by $x_1', x_2', ..., x_k'$, respectively. And since $x_1 + x_1' + x_2 + x_2' + ... + x_k^c + x_k' = N_1 + 1 + N_2$, we must have that $x_1' + x_2' + ... + x_k' = N_2$. At this time since $(x_1 + x_1', x_2 + x_2', ..., x_k^c + x_k')$ is a potential target point, we locate the biggest and the second biggest value from them, i.e., $x_a'(k) = x_a + x_a' = max\{x_k^c + x_k', x_{k-1} + x_{k-1}', ..., x_1 + x_1'\}$ and $x_b'(k-1) = x_b + x_b' = secondmax\{x_k^c + x_k', x_{k-1} + x_{k-1}', ..., x_1 + x_1'\}$, where a and b represent the indices of the biggest and the second biggest numbers in categories $1, 2, ..., k$, respectively. Since $(x_1 + x_1', x_2 + x_2', ..., x_k^c + x_k')$ is a potential target point, we have that $x_a'(k) - x_b'(k-1) = D_2 - 1$. And since $0 \le x_1 + x_1' \le M_1$, $0 \le x_2 + x_2' \le M_2$, ..., and $0 \le x_k^c + x_k' \le M_k$, we must also have

that $0 \le x'_1 \le M_1 - x_1$, $0 \le x'_2 \le M_2 - x_2$, ..., and $0 \le x'_k \le M_k - x^c_k$. The target probability associated with the potential target point $(x_1 + x'_1, x_2 + x'_2, ..., x^c_k + x'_k)$ is then given by

$$\frac{\binom{M_1-x_1}{x'_1}\binom{M_2-x_2}{x'_2}...\binom{M_k-x^c_k}{x'_k}}{\binom{N-x_1-...-x^c_k}{N_2}} \frac{M_a - x_a - x'_a}{N - x_1 - ... - x^c_k - x'_1 - ... - x'_k}.$$

Therefore the target probability in layer $N_1 + 1 + N_2$ transfered from the potential target point $(x_1, x_2, ..., x_k)$ is given by

$$\sum_{\substack{x^c_k=x_k+1 \\ x'_1+x'_2+...+x'_k=N_2 \\ x'_a(k)=x_a+x'_a=max\{x^c_k+x'_k,x_{k-1}+x'_{k-1},...,x_1+x'_1\} \\ x'_b(k-1)=x_b+x'_b=secondmax\{x^c_k+x'_k,x_{k-1}+x'_{k-1},...,x_1+x'_1\} \\ x'_a(k)-x'_b(k-1)=D_2-1}} \sum_{x'_k=0}^{M_k-x^c_k} \sum_{x'_{k-1}=0}^{M_{k-1}-x_{k-1}} \cdots \sum_{x'_1=0}^{M_1-x_1} \frac{\binom{M_1-x_1}{x'_1}\binom{M_2-x_2}{x'_2}...\binom{M_k-x^c_k}{x'_k}}{\binom{N-x_1-...-x^c_k}{N_2}} \frac{M_a-x_a-x'_a}{N-x_1-...-x^c_k-x'_1-...-x'_k}$$

Summing all of the target probabilities in layer N_1+1+N_2 transfered from all the potential target points in layer N_1 gives (3.2). (3.3) is proved similarly.

A.1.4 Proof of Theorem 3.9

Since the condition of Procedure 2 is $X_{(k)t} - X_{(k-1)t} \ge D_2$ with $D_2 > 1$, we must have that $X_{(k)t} > 0$ and $X_{(k-1)t} \ge 0$. The first target point that can be reached under Procedure 2 is $X_{1t} = 0, X_{2t} = 0, ..., X_{kt} = D_2$. The corresponding potential target point is $(X_{1t}, X_{2t}, ..., X_{kt}) = (0, 0, ..., D_2 - 1)$. Therefore the first layer that can result in rejecting H_0 under Procedure 2 is layer $D_2 - 1$. Since the sampling time $t \le N_0$, the last possible layer is layer $N_0 - 1$. Let $P_n = Pr(\widetilde{\bigcup}_{D_2-1 \le i < n} S_i)$, where S_i is the target event set at layer i. Then $P_{H_0}(X_{(k)t} - X_{(k-1)t} \ge D_2) = P_{N_0}$.

Let us prove that

$$P_n = \sum_{i=D_2-1}^{n-1} C_i - \sum_{\substack{D_2-1\leq i_1<i_2<n \\ i_2>i_1+1}} C_{i_1,i_2}$$

$$+ ... + (-1)^{r+1} \sum_{\substack{D_2-1\leq i_1<i_2<...<i_r<n \\ i_2>i_1+1, i_3>i_2+1,...,i_r>i_{r-1}+1 \\ r\leq \frac{n-D_2}{2}+1}} C_{i_1,i_2,...,i_r} \qquad (A.1)$$

by induction.

Base case: From $(A.1)$ with $n = D_2 - 1$ we see that $P_{D_2} = C_{D_2-1}$. On the other hand $P_{D_2} = Pr(\widetilde{\bigcup}_{D_2-1\leq i<D_2} S_i) = Pr(S_{D_2-1}) = C_{D_2-1}$. So the base case is proved.

Induction step:

Suppose that

$$P_m = \sum_{i=D_2-1}^{m-1} C_i - \sum_{\substack{D_2-1\leq i_1<i_2<m \\ i_2>i_1+1}} C_{i_1,i_2}$$

$$+ ... + (-1)^{r+1} \sum_{\substack{D_2-1\leq i_1<i_2<...<i_r<m \\ i_2>i_1+1, i_3>i_2+1,...,i_r>i_{r-1}+1 \\ r\leq \frac{m-D_2}{2}+1}} C_{i_1,i_2,...,i_r}$$

Then

$$\begin{aligned} P_{m+1} &= Pr((\widetilde{\bigcup}_{D_2-1\leq i<m} S_i)\widetilde{\bigcup} S_m) \\ &= Pr((\widetilde{\bigcup}_{D_2-1\leq i<m} S_i)) + Pr(S_m) - Pr((\widetilde{\bigcup}_{D_2-1\leq i<m} S_i)\widetilde{\bigcap} S_m) \\ &= P_m + Pr(S_m) - Pr((\widetilde{\bigcup}_{D_2-1\leq i<m} S_i)\widetilde{\bigcap} S_m) \end{aligned}$$

According to Definition 3.8, we get that

$$\widetilde{\bigcup}_{D_2-1\le i<m} S_i = (\bigcup_{D_2-1\le i<m} S_i) \setminus (\bigcup_{D_2-1\le i_1<i_2<m} S_{i_1} \tilde{\cap} S_{i_2}) \bigcup (\bigcup_{D_2-1\le i_1<i_2<i_3<m} S_{i_1} \tilde{\cap} S_{i_2} \tilde{\cap} S_{i_3}) \setminus ... \bigcup ...$$

Therefore

$$\begin{aligned}
P_{m+1} = P_m + Pr(S_m) - \\
-Pr(\{(\bigcup_{D_2-1\le i<m} S_i) \setminus (\bigcup_{D_2-1\le i_1<i_2<m} S_{i_1} \tilde{\cap} S_{i_2}) \bigcup (\bigcup_{D_2-1\le i_1<i_2<i_3<m} S_{i_1} \tilde{\cap} S_{i_2} \tilde{\cap} S_{i_3}) \setminus ...\} \widetilde{\bigcap} S_m) \\
= P_m + Pr(S_m) - \\
-Pr((\bigcup_{D_2-1\le i<m} S_i \tilde{\cap} S_m) \setminus (\bigcup_{D_2-1\le i_1<i_2<m} S_{i_1} \tilde{\cap} S_{i_2} \tilde{\cap} S_m) \bigcup (\bigcup_{D_2-1\le i_1<i_2<i_3<m} S_{i_1} \tilde{\cap} S_{i_2} \tilde{\cap} S_{i_3} \tilde{\cap} S_m) \setminus ...) \\
= \sum_{i=D_2-1}^{m-1} C_i - \sum_{\substack{D_2-1\le i_1<i_2<m \\ i_2>i_1+1}} C_{i_1,i_2} \\
+ ... + (-1)^{r+1} \sum_{\substack{D_2-1\le i_1<i_2<...<i_r<m \\ i_2>i_1+1, i_3>i_2+1, ..., i_r>i_{r-1}+1 \\ r\le \frac{m-D_2}{2}+1}} C_{i_1,i_2,...,i_r}
\end{aligned}$$

$$+C_m - \sum_{i=D_2-1}^{m-1} C_{i,m} + \sum_{\substack{D_2-1\leq i_1<i_2<m \\ i_2>i_1+1}} C_{i_1,i_2,m}$$

$$+...+(-1)^r \sum_{\substack{D_2-1\leq i_1<i_2<...<i_r<m \\ i_2>i_1+1,i_3>i_2+1,...,i_r>i_{r-1}+1 \\ r\leq\frac{m-D_2}{2}+1}} C_{i_1,i_2,...,i_r,m}$$

$$= \sum_{i=D_2-1}^{m} C_i - \sum_{\substack{D_2-1\leq i_1<i_2<m+1 \\ i_2>i_1+1}} C_{i_1,i_2}$$

$$+...+(-1)^{r+1} \sum_{\substack{D_2-1\leq i_1<i_2<...<i_r<m+1 \\ i_2>i_1+1,i_3>i_2+1,...,i_r>i_{r-1}+1 \\ r\leq\frac{m+1-D_2}{2}+1}} C_{i_1,i_2,...,i_r}$$

And since layer N_1 can transfer into layer N_2 such that $N_2 \geq N_1+2$, then $D_2-1+2(r-1) \leq N_0-1$.We can get that $r \leq \frac{N_0-D_2}{2}+1$.

So the theorem is proved.

A.1.5 Fast Algorithm of C_{N_1,N_1+1+N_2}s

Calculating just a single transfer probability requires a heavy computational effort and time. It needs too much time in the calculation for all the transfer probabilities. But there are plenty of common parts in the calculation of the transfer probabilities. Therefore when we calculate the transfer probabilities, lots of time is spent doing the same calculation repeatedly. If we use the results of common parts for all the transfer probabilities, it will save lots of calculation and time. We definitely can calculate all the transfer probabilities in one big structure. Because this work is too complex and needs more time working on it, it is impossible for us to do this job currently, so we just leave it for a future research. Here we just describe the simple case.

C_{N_1,N_1+1+N_2} summates the transfer probabilities from all the potential target points in layer N_1 to those in layer N_1+1+N_2. The structure of the transfer probability algorithm is a tree. The depth of the layers directly affect the speed of the calculations for the transfer probability. We may calculate different transfer probabilities from the same layer to different layers together to save time. For example we can calculate C_{N_1,N_1+1+N_2} and $C_{N_1,N_1+1+N_2'}$ together. Since the root of C_{N_1,N_1+1+N_2} is C_{N_1}, we first take a potential target point in N_1 and calculate its target probability and then search all the potential target points in layer N_1+1+N_2 transfered from this potential target point in layer N_1 and calculate the target probabilities of all these potential target points in layer N_1+1+N_2. At the same time we also can calculate the target probabilities of all the potential target points in layer N_1+1+N_2' transfered from the same potential target point in layer N_1. The algorithm is as follows:

Let

$$D_{(x_1,x_2,\ldots,x_k^c)\to N_1+1+N_2} = \sum_{\substack{x_k'=0 \\ x_k^c=x_k+1 \\ x_1'+x_2'+\ldots+x_k'=N_2 \\ x_a'(k)=x_a+x_a'=max\{x_k^c+x_k',x_{k-1}+x_{k-1}',\ldots,x_1+x_1'\} \\ x_b'(k-1)=x_b+x_b'=secondmax\{x_k^c+x_k',x_{k-1}+x_{k-1}',\ldots,x_1+x_1'\} \\ x_a'(k)-x_b'(k-1)=D_2-1}}^{M_k-x_k^c} \sum_{x_{k-1}'=0}^{M_{k-1}-x_{k-1}} \cdots \sum_{x_1'=0}^{M_1-x_1} \frac{\binom{M_1-x_1}{x_1'}\binom{M_2-x_2}{x_2'}\cdots\binom{M_k-x_k^c}{x_k'}}{\binom{N-x_1-\ldots-x_k^c}{N_2}} \frac{M_a-x_a-x_a'}{N-x_1-\ldots-x_k^c-x_1'-\ldots-x_k'}\}$$

$$\begin{pmatrix} C_{N_1,N_1+1+N_2} \\ C_{N_1,N_1+1+N_2'} \\ C_{N_1,N_1+1+N_2''} \\ \cdots\cdots \end{pmatrix} = k\{ \sum_{x_{k-1}=0}^{N_1-D_2+1} \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} \cdots \sum_{x_1=0}^{x_{k-1}-1}$$

$$x_k = x_{k-1} + D_2 - 1$$

$$x_{k-1} = x_{k-2} = \ldots = x_{k-j}$$

$$x_1 + x_2 + \ldots + x_k = N_1$$

$$B(x_k, x_{k-1}, \ldots, x_1) \begin{pmatrix} D_{(x_1,x_2,\ldots,x_k^c)\to N_1+1+N_2} \\ D_{(x_1,x_2,\ldots,x_k^c)\to N_1+1+N_2'} \\ D_{(x_1,x_2,\ldots,x_k^c)\to N_1+1+N_2''} \\ \cdots\cdots \end{pmatrix}$$

For the transfer probabilities whose depth is more than 2, we can get similar efficient algorithms.

A.1.6 An Example

In order to understand and explain the concepts, theorems and their relationships clearly in chapter 3, here we give an example of $k = 3$. Although the example is still very simple, we can see the main ideas of the concepts in chapter 3.

Suppose that $k = 3, M_1 = 8, M_2 = 8, M_3 = 8, D_2 = 2$, and $N_0 = 6$. Therefore $N = M_1 + M_2 + M_3 = 24$. Let us denote an object from category 1 by A, category 2 by B, and category 3 by C. The potential target points are then given by

Table A.1.1 The potential target points

Layer	Category 1	Category 2	Category 3
1	0	0	1
	0	1	0
	1	0	0
3	0	1	2
	1	0	2
	0	2	1
	1	2	0
	2	0	1
	2	1	0
4	1	1	2
	1	2	1
	2	1	1
5	0	2	3
	2	0	3
	0	3	2
	2	3	0
	3	0	2
	3	2	0

From layer 1 of potential target points in the above table we can immediately get the target events AA, BB, CC. From the first line in layer 3 we know that the target events of this point have 1 B and 3 C, so we immediately get the target events of this potential target point is $BCCC$, $CBCC$, $CCBC$. But $CCCB$ is not a target event, since it is transfered from $W_{kt} = 3$ into $W_{kt} = 2$.

The target probability of all target events are given as follows:

Note: "# events" is the number of the same type of target events with the same probability, so we just give the target probability of the first event of the same type.

Table A.1.2 The target events with their target probabilities and transfer probabilities

layer	target event	target probability	# events	probability type
1	CC	8/24*7/23	3	C_1
	BB			C_1
	AA			C_1
3	BCCC	8/24*8/23*7/22*6/21	12	C_3
	CBCC			C_3
	CCBC	8/24*7/23*8/22*6/21	6	$C_{1,3}$ and C_3
	ACCC			C_3
	CACC			C_3
	CCAC			$C_{1,3}$ and C_3
	CBBB			C_3
	BCBB			C_3
	BBCB			$C_{1,3}$ and C_3
	ABBB			C_3
	BABB			C_3
	BBAB			$C_{1,3}$ and C_3
	CAAA			C_3
	ACAA			C_3
	AACA			$C_{1,3}$ and C_3
	BAAA			C_3
	ABAA			C_3
	AABA			$C_{1,3}$ and C_3

layer	target event	target probability	# events	probability type
4	ABCCC	8/24*8/23*8/22*7/21*6/20	30	C_4
	BACCC			C_4
	CABCC			C_4
	CBACC			C_4
	CCABC	8/24*7/23*8/22*8/21*6/20	6	$C_{1,4}$ and C_4
	CCBAC			$C_{1,4}$ and C_4
	ACBCC			C_4
	BCACC			C_4
	CACBC			C_4
	CBCAC			C_4
	ACCBC			C_4
	BCCAC			C_4
	ACBBB			C_4
	CABBB			C_4
	BACBB			C_4
	BCABB			C_4
	BBACB			$C_{1,4}$ and C_4
	BBCAB			$C_{1,4}$ and C_4
	ABCBB			C_4
	CBABB			C_4
	BABCB			C_4
	BCBAB			C_4
	ABBCB			C_4
	CBBAB			C_4
	BCAAA			C_4
	CBAAA			C_4
	ABCAA			C_4
	ACBAA			C_4
	AABCA			$C_{1,4}$ and C_4
	AACBA			$C_{1,4}$ and C_4
	BACAA			C_4
	CABAA			C_4
	ABACA			C_4
	ACABA			C_4
	BAACA			C_4
	CAABA			C_4

layer	target event	target probability	# events	probability type
5	BBCCCC	8/24*7/23*6/22*8/21*7/20*5/19	18	$C_{1,5}$ and C_5
	CBBCCC	8/24*7/23*8/22*7/21*6/20*5/19	24	C_5
	CCBBCC			$C_{1,5}$ and C_5
	CCCBBC			$C_{1,5}$ and C_5
	BCBCCC			C_5
	BCCBCC			C_5
	BCCCBC	8/24*7/23*8/22*7/21*6/20*5/19	12	$C_{3,5}$ and C_5
	CBCBCC			C_5
	CBCCBC			$C_{3,5}$ and C_5
	CCBCBC	8/24*7/23*8/22*7/21*6/20*5/19	6	$C_{1,3,5}$, $C_{1,5}$, $C_{3,5}$ and C_5
	AACCCC			$C_{1,5}$ and C_5
	CAACCC			C_5
	CCAACC			$C_{1,5}$ and C_5
	CCCAAC			$C_{1,5}$ and C_5
	ACACCC			C_5
	ACCACC			C_5
	ACCCAC			$C_{3,5}$ and C_5
	CACACC			C_5
	CACCAC			$C_{3,5}$ and C_5
	CCACAC			$C_{1,3,5}$, $C_{1,5}$, $C_{3,5}$ and C_5
	CCBBBB			$C_{1,5}$ and C_5
	BCCBBB			C_5
	BBCCBB			$C_{1,5}$ and C_5
	BBBCCB			$C_{1,5}$ and C_5
	CBCBBB			C_5
	CBBCBB			C_5
	CBBBCB			$C_{3,5}$ and C_5
	BCBCBB			C_5
	BCBBCB			$C_{3,5}$ and C_5
	BBCBCB			$C_{1,3,5}$, $C_{1,5}$, $C_{3,5}$ and C_5
	AABBBB			$C_{1,5}$ and C_5
	BAABBB			C_5
	BBAABB			$C_{1,5}$ and C_5
	BBBAAB			$C_{1,5}$ and C_5
	ABABBB			C_5
	ABBABB			C_5
	ABBBAB			$C_{3,5}$ and C_5
	BABABB			C_5
	BABBAB			$C_{3,5}$ and C_5
	BBABAB			$C_{1,3,5}$, $C_{1,5}$, $C_{3,5}$ and C_5

layer	target event	target probability	# events	probability type
5	CCAAAA			$C_{1,5}$ and C_5
	ACCAAA			C_5
	AACCAA			$C_{1,5}$ and C_5
	AAACCA			$C_{1,5}$ and C_5
	CACAAA			C_5
	CAACAA			C_5
	CAAACA			$C_{3,5}$ and C_5
	ACACAA			C_5
	ACAACA			$C_{3,5}$ and C_5
	AACACA			$C_{1,3,5}$, $C_{1,5}$, $C_{3,5}$ and C_5
	BBAAAA			$C_{1,5}$ and C_5
	ABBAAA			C_5
	AABBAA			$C_{1,5}$ and C_5
	AAABBA			$C_{1,5}$ and C_5
	BABAAA			C_5
	BAABAA			C_5
	BAAABA			$C_{3,5}$ and C_5
	ABABAA			C_5
	ABAABA			$C_{3,5}$ and C_5
	AABABA			$C_{1,3,5}$, $C_{1,5}$, $C_{3,5}$ and C_5

Using the above table, if we just count the target probabilities that are not also transfer probabilities then we get:

$$\begin{aligned} P(W_{kt} \geq D2) &= 8/24 * 7/23 * 3 + 8/24 * 8/23 * 7/22 * 6/21 * 12 \\ &+8/24 * 8/23 * 8/22 * 7/21 * 6/20 * 30 + \\ &+8/24 * 7/23 * 8/22 * 7/21 * 6/20 * 5/19 * 24 \\ &= 0.5806116 \end{aligned}$$

Now we use theorem 3.9 to calculate $P(W_{kt} \geq D2)$:

$$
\begin{aligned}
C_1 &= 8/24 * 7/23 * 3 = 0.3043478 \\
C_3 &= 8/24 * 8/23 * 7/22 * 6/21 * 18 = 0.1897233 \\
C_4 &= 8/24 * 8/23 * 8/22 * 7/21 * 6/20 * 36 = 0.1517787 \\
C_5 &= 8/24 * 7/23 * 6/22 * 8/21 * 7/20 * 5/19 * 60 = 0.05824839 \\
C_{1,3} &= 8/24 * 7/23 * 8/22 * 6/21 * 6 = 0.06324111 \\
C_{1,4} &= 8/24 * 7/23 * 8/22 * 8/21 * 6/20 * 6 = 0.02529644 \\
C_{1,5} &= 8/24 * 7/23 * 8/22 * 7/21 * 6/20 * 5/19 * 24 = 0.02329936 \\
C_{3,5} &= 8/24 * 7/23 * 8/22 * 7/21 * 6/20 * 5/19 * 18 = 0.01747452 \\
C_{1,3,5} &= 8/24 * 7/23 * 8/22 * 7/21 * 6/20 * 5/19 * 6 = 0.005824839
\end{aligned}
$$

Therefore

$$
P(W_{kt} \geq D2) = C_1 + C_3 + C_4 + C_5 - C_{1,3} - C_{1,4} - C_{1,5} - C_{3,5} + C_{1,3,5} = 0.5806116
$$

We used Theorem 3.5, 3.7 and 3.9 to make a program to calculate $P(W_{kt} \geq D2)$. The

results of running our program for the above example are as follows:

$$
\begin{aligned}
P(W_{kt} \geq D_2) &= 0.580612 \\
C_1 &= 0.304348 \\
C_3 &= 0.189723 \\
C_4 &= 0.151779 \\
C_5 &= 0.058248 \\
C_{1,3} &= 0.063241 \\
C_{1,4} &= 0.025296 \\
C_{1,5} &= 0.023299 \\
C_{3,5} &= 0.017475 \\
C_{1,3,5} &= 0.005825 \\
C_{2,4} &= 0.0 \\
C_{2,5} &= 0.0
\end{aligned}
$$

So using the formulae in Theorem 3.5, 3.7 and 3.9 we get the exact results in this example.

A.2 Proofs from Chapter 4

A.2.1 Proof of Theorem 4.1

Under the alternative hypothesis H_a: $M_1 = M_2 = ... = M_{k-1}$ and $M_k = \lambda M_1$. We can take $X_{(k)} = X_{kt}$ and $X_{(k)} = X_{1t}$ to be the random variables with the biggest values. For $X_{(k)} = X_{kt}$, since $M_1 = M_2 = ... = M_{k-1}$, we can take $X_{k-1,t}$ as the second largest variable. Following the proof of theorem 3.5, we can get the formula of this case:

$$\sum_{\substack{x_{k-1}=0 \\ x_k=x_{k-1}+D_2-1 \\ x_{k-1}=x_{k-2}=...=x_{k-j} \\ x_1+x_2+...+x_k=N_1}}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=1}^{k-1} \binom{k-1}{j} \sum_{x_{k-j-1}=0}^{x_{k-1}-1} \sum_{x_{k-j-2}=0}^{x_{k-1}-1} \cdots \sum_{x_1=0}^{x_{k-1}-1} B_k(x_1, ..., x_{k-1}, x_k),$$

where $B_k(x_1, ..., x_{k-1}, x_k)$ is the target probability of the potential target point $(x_1, ..., x_{k-1}, x_k)$ becoming the target point $(x_1, ..., x_{k-1}, x_k + 1)$.

For $X_{(k)} = X_{1t}$, we can take $X_{(k-1)} = X_{2t}$ or $X_{(k-1)} = X_{kt}$ as the second largest variable. For $X_{(k-1)} = X_{2t}$, we can have j variables with identical values such that $x_2 = x_3 = ... = x_{1+j}$, $j = 1, 2, ..., k-2$. In this situation we assume that $x_2 > x_k$. Therefore, the target probability of all the potential target points of this subset in layer N_1 is given by

$$\sum_{\substack{x_2=0 \\ x_1=x_2+D_2-1 \\ x_2=x_3=...=x_{1+j} \\ x_1+x_2+...+x_k=N_1}}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=1}^{k-2} \binom{k-2}{j} \sum_{x_{2+j}=0}^{x_2-1} \sum_{x_{3+j}=0}^{x_2-1} \cdots \sum_{x_k=0}^{x_2-1} B_1(x_1, ..., x_{k-1}, x_k),$$

where $B_1(x_1, ..., x_{k-1}, x_k)$ is the target probability of the potential target point $(x_1, ..., x_{k-1}, x_k)$ becoming the target point $(x_1 + 1, ..., x_{k-1}, x_k)$.

For $X_{(k)} = X_{1t}$, $X_{(k-1)} = X_{kt}$, we can have $(j+1)$ variables with identical values such that $x_k = x_{k-1} = ... = x_{k-j}$, $j = 0, 2, ..., k-2$. At this situation we suppose that x_2 can be equal to x_k. Therefore, the target probability of all the potential target points in this subset in layer N_1 is given by

$$\sum_{\substack{x_k=0 \\ x_1=x_k+D_2-1 \\ x_k=x_{k-1}=...=x_{k-j} \\ x_1+x_2+...+x_k=N_1}}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=0}^{k-2} \binom{k-2}{j} \sum_{x_{k-j-1}=0}^{x_k-1} \sum_{x_{k-j-2}=0}^{x_k-1} \cdots \sum_{x_2=0}^{x_k-1} B_1(x_1, ..., x_{k-1}, x_k)$$

So adding the above two formulae together, for $X_{(k)} = X_{1t}$, the target probability of the

potential target points of this subset in layer N_1 is given by

$$\sum_{x_2=0}^{\frac{1}{2}(N_1-D_2+1)} \sum_{\substack{j=1 \\ x_1=x_2+D_2-1 \\ x_2=x_3=...=x_{1+j} \\ x_1+x_2+...+x_k=N_1}}^{k-2} \binom{k-2}{j} \sum_{x_{2+j}=0}^{x_2-1} \sum_{x_{3+j}=0}^{x_2-1} \cdots \sum_{x_k=0}^{x_2-1} B_1(x_1, ..., x_{k-1}, x_k)$$

$$+ \sum_{\substack{x_k=0 \\ x_1=x_k+D_2-1 \\ x_k=x_{k-1}=...=x_{k-j} \\ x_1+x_2+...+x_k=N_1}}^{\frac{1}{2}(N_1-D_2+1)} \sum_{j=0}^{k-2} \binom{k-2}{j} \sum_{x_{k-j-1}=0}^{x_k-1} \sum_{x_{k-j-2}=0}^{x_k-1} \cdots \sum_{x_2=0}^{x_k-1} B_1(x_1, ..., x_{k-1}, x_k)$$

And since each of $X_{1t}, X_{2t}, ..., X_{k-1,t}$ have the same probability of being $X_{(k)}$, the target probability in layer N_1 under H_a is given by the formula in Theorem 4.1.

A.3 Fast Algorithm for Multiple Summation

We need to find the ways to find all the values of $x_1, x_2, ..., x_k$ satisfying $x_1+x_2+...+x_k = n$ since there are k degrees of freedom under the condition of $x_1+x_2+...+x_k = n$, i.e., all k variables $x_1, x_2, ..., x_k$ are freely changeable. One simple algorithm would be to give all the possible values of $x_1, x_2, ..., x_k$ to find the values satisfying $x_1+x_2+...+x_k = n$. This method is inefficient since too many non-used x_i's are produced; It is therefore necessary to find a fast algorithm to quickly target the required x_is satisfying $x_1+x_2+...+x_k = n$. Since $\sum_{i=1}^{k} x_i = n$, we must have that $\forall i \in \{1, 2, ..., k\}, x_i \leq n$, we can use "nesting for loops" on every $x_i = 1(1)n$ like the R program "$for(x_1\ in\ 1:n)\{for(x_2\ in\ 1:n)\{...for(x_k\ in\ 1:n)\{\#the\ other\ codes.\}...\}\}$". But it is not flexible while k is changing. A very useful and flexible alternative way is that $x_1, x_2, ..., x_k$ can be regarded as a k digit $(n+1)$-nary number, $x_1(n+1)^{k-1} + x_2(n+1)^{k-2} + ... + x_k$. Any bit x_i reaching a count over n will be carried onto the bit x_{i-1}, i.e., when x_i reaches $n+1$, then x_i becomes 0, and x_{i-1} is increased by 1. For example in a decimal number system, let $x_1 = 0$ and $x_2 = 9$. Then $x_1x_2 = 09$ represents 9. Suppose that we now add 1 to x_2. Then since $x_2+1 = 10$ is over 9 by 1, the 1 is carried from x_2 into x_1, so that $x_1x_2 = 10$. That means

$x_1 = 1$, $x_2 = 0$. Using this idea the basic algorithm for multiple summation is as follows. Note: all the algorithms below (except algorithm $A.3.6$) start with $x_1 = 0, x_2 = 0, ..., x_k = 0$ and stop when $x_1 = n + 1$.

Basic algorithm A.3.1: Suppose the current numbers are $x_1, x_2, ..., x_k$. We obtain the next values $x'_1, x'_2, ..., x'_k$ by increasing x_k by 1. If $x'_k > n$, then we set $x'_k = 0$ and $x'_{k-1} = x_{k-1} + 1$, similarly, Let $i = k - 1, ..., 3, 2$ in sequence, if $x'_i > n$, then we set $x'_i = 0$ and $x'_{i-1} = x_{i-1} + 1$. Then we check $x'_1 + x'_2 + ... + x'_k$. If $x'_1 + x'_2 + ... + x'_k = n$, then we find the number next to $x_1, x_2, ..., x_k$. If $x'_1 + x'_2 + ... + x'_k \neq n$, then continue the above procedure from the start point.

For example in a 4-nary system, suppose $k = 3$ and the current numbers are $x_1 = 0, x_2 = 0, x_3 = 2$. Since $x'_3 = x_3 + 1 = 3$, then the next numbers become $x'_1 = 0, x'_2 = 0, x'_3 = 3$. Suppose the current numbers are $x_1 = 0, x_2 = 0, x_3 = 3$. $x'_3 = x_3 + 1 = 4 > 3$, there is a carry from x_3 to x_2, so the next numbers become $x'_1 = 0, x'_2 = 1, x'_3 = 0$. Suppose the current numbers are $x_1 = 1, x_2 = 3, x_3 = 3$. $x'_3 = x_3 + 1 = 4 > 3$, there is a carry from x_3 to x_2, so $x'_3 = 0, x'_2 = x_2 + 1 = 4 > 3$, there is a carry from x'_2 to x'_1, so $x'_2 = 0, x'_1 = x_1 + 1 = 2$. so the next numbers become $x'_1 = 2, x'_2 = 0, x'_3 = 0$.

In order to find a group of values of $x_1, x_2, ..., x_k$ satisfying $x_1 + x_2 + ... + x_k = n$, it takes too many procedures in the above basic algorithm. It is not an effective way for finding the values of $x_1, x_2, ..., x_k$. We need to find more powerful algorithms to do this job and hence save time. We introduce a theorem first.

Theorem A.3.2: Suppose $x_1, x_2, ..., x_k$ are bits of a $(n + 1)$-nary number η, i.e., $\eta = x_1(n + 1)^{k-1} + x_2(n + 1)^{k-2} + ... + x_k$ with $0 \leq x_i \leq n$, $\forall i = 1, 2, ..., k$. Let $S = x_1 + x_2 + ... + x_k$. If a number $\omega = o_1(n + 1)^{k-1} + o_2(n + 1)^{k-2} + ... + o_k$ is added to η producing μ carries, then the new bits of $\eta + \omega$, $x'_1, x'_2, ..., x'_k$, satisfy $x'_1 + x'_2 + ... + x'_k = S + o_1 + o_2 + ... + o_k - \mu n$, where $0 \leq o_i \leq n$, $\forall i = 1, 2, ..., k$.

Proof. Suppose $\exists i \in \{1, 2, ..., k\}$, there is no carry on x'_i from x'_{i+1} (If there is a carry on x'_i from x'_{i+1}, the proof is similar just let $x'_i = x'_i + 1$ first.). But $x_i + o_i > n$, so there is a carry from x'_i into x'_{i-1}, and after the carry, $x'_i = x_i + o_i - (n+1)$ and $x'_{i-1} = x_{i-1} + o_{i-1} + 1$. We have that $x'_{i-1} + x'_i = x_{i-1} + x_i + o_{i-1} + o_i - n$. So for any one carry from x'_i, x'_i will lose $n+1$ and x'_{i-1} will gain 1. Similarly at this time if $x_{i-1} + o_{i-1} + 1 > n$, then $x'_{i-1} = x_{i-1} + o_{i-1} + 1 - (n+1)$ and $x'_{i-2} = x_{i-2} + o_{i-2} + 1$, so $x'_{i-2} + x'_{i-1} + x'_i = x_{i-2} + x_{i-1} + x_i + o_{i-2} + o_{i-1} + o_i - 2n$. Continuing this process, we see that if there is μ carries, then the bit summation of $(\eta + \omega)$ will be $x'_1 + x'_2 + ... + x'_k = S + o_1 + o_2 + ... + o_k - \mu n$.

For example in 5-nary system, suppose $k = 3$ and the current number is $x_1x_2x_3$ with bits $x_1 = 1, x_2 = 2, x_3 = 4$. We add $o_1o_2o_3$ with $o_1 = 1, o_2 = 2, o_3 = 2$ to $x_1x_2x_3$. Since $x'_3 = x_3 + o_3 = 6 > 4$, there is 1 carry from x'_3 to x'_2. Also, $x'_2 = x_2 + o_2 + 1 = 5 > 4$, so there is a carry from x'_2 to x'_1. In total there are two carries in this addition. So, according to Theorem $A.3.2$, $x'_1 + x'_2 + x'_3 = x_1 + x_2 + x_3 + o_1 + o_2 + o_3 - 2 \times 4 = 12 - 8 = 4$. As a check, note that after the addition, $x'_1 = 3, x'_2 = 0, x'_3 = 1$, so $x'_1 + x'_2 + x'_3 = 4$.

Fast algorithm A.3.3: Suppose $x_1, x_2, ..., x_k$ are bits of one of the ordered k digit $(n+1)$-nary numbers in increasing order of magnitude whose summation of bits are n, i.e., $x_1 + x_2 + ... + x_k = n$. Let $\eta = x_1(n+1)^{k-1} + ... + x_k$ and let the bits of the next $(n+1)$-nary number after η whose bits add to n be $x'_1, x'_2, ..., x'_k$. Then the procedure for finding $x'_1, x'_2, ..., x'_k$, given $x_1, x_2, ..., x_k$ is as follows:

Let $x'_k = x_k + n$ and if $x'_k > n$, then $x'_k = x_k - 1$, $x'_{k-1} = x_{k-1} + 1$. For $i \in \{2, 3, ..., k-1\}$, if $x'_i > n$, then $x'_i = 0$ and $x'_{i-1} = x_{i-1} + 1$. If $x'_1 + x'_2 + ... + x'_k = n$, then we have the next number after η whose bits add to n. If $x'_1 + x'_2 + ... + x'_k \neq n$, let $x''_k = x'_k + n$ and continue the above procedure.

Proof. Suppose $x_k > 0$, then $x'_k = x_k + n > n$. There is a carry from x'_k into x'_{k-1}. There isn't any other carry since $x_1 + x_2 + ... + x_{k-1} < n$. According to Theorem $A.3.2$ $x'_1 + x'_2 + ... + x'_k = x_1 + x_2 + ... + x_k + n - n = n$. So after just one step we have already found the bits of another number satisfying $\sum_{i=1}^{k} x'_i = n$. Furthermore $x'_1, x'_2, ..., x'_k$ are the bits of the next number after η whose summation is n since if we add a number n' to η such that $n' < n$, then according to Theorem $A.3.2$ $x'_1 + x'_2 + ... + x'_k = n + n' - n = n' \neq n$.

Suppose $x_k = 0$ and $\exists j$ such that $x_{j+1} = x_{j+2} = ... = x_k = 0$. Let $\eta = x_1(n+1)^{k-1} + x_2(n+1)^{k-2} + ... + x_j(n+1)^{k-j}$ whose bits add to n and let $\omega = n(n+1)^{k-j-1} + n(n+1)^{k-j-2} + ... + n$. Clearly ω is multiple of n, which can be obtained by repeatedly adding n to x_k. And since there is no carry in this addition, then according to Theorem $A.3.2$, the summation of bits of $\eta + \omega$ is $n + (k-j)n$. We then add another $\omega' = n$, i.e., $x'_k = x_k + n$. Then there are $k - j$ carries, so according to theorem $A.3.2$, the summation of bits of $\eta + \omega + \omega'$ is $n + (k-j)n + n - (k-j)n = 2n$. The bits of $\eta + \omega + \omega'$ then become $x_1, x_2, ..., (x_j + 1), 0, 0, ..., 0, n - 1$. If $x_j = n$, then there is a carry from x_j into x_{j-1}, and therefore the summation of bits of $\eta + \omega + \omega'$ will become n again, so the next value $x'_1 + x'_2 + ... + x'_k = n$ is found. If $x_j + 1 \leq n$, then we add ω $(n - x_j + 1)$ times. Since each time the bits after x_j will carry one by one starting at x_k into x_j, according to Theorem $A.3.2$, each time before x_j carry into x_{j-1} that $x'_1 + x'_2 + ... + x'_k = 2n$. After the last addition of ω into η, x_j carries into x_{j-1}, so $x'_1 + x'_2 + ... + x'_k = n$ again. So $x'_1, x'_2, ..., x'_k$ are found. Since $x_1 + x_2 + ... + x_j = n$, before x_j carries into x_{j-1}, $x'_1 + x'_2 + ... + x'_j \geq n$, the bits $x'_1, x'_2, ..., x'_k$ that include the first carry from x'_j into x'_{j-1} are the bits of the next number after η.

Because in this algorithm, each step $x'_k = x_k + n$ and in basic algorithm $A.3.1$, each step $x'_k = x_k + 1$, it is faster than the basic algorithm.

For example, suppose that $k = 5$, $n = 7$, and $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 2, 1, 4)$. It is clear that $x_1 + x_2 + x_3 + x_4 + x_5 = 7$. According to algorithm $A.3.3$, Let $x'_5 = x_5 + 7 = 11 > 7$. So $x'_5 = x_5 - 1 = 3$. $x'_4 = x_4 + 1 = 2$. $x'_1 = x_1 = 0$, $x'_2 = x_2 = 0$, $x'_3 = x_3 = 2$. So

$(x'_1, x'_2, x'_3, x'_4, x'_5) = (0, 0, 2, 2, 3)$ and $x'_1 + x'_2 + ... + x'_5 = 7$.

If $(x_1, x_2, x_3, x_4, x_5) = (1, 1, 5, 0, 0)$, then we change $(x_1, x_2, x_3, x_4, x_5)$ to $(x'_1, x'_2, x'_3, x'_4, x'_5) = (1, 1, 5, 7, 7)$ by repeatedly adding 7 to x_5. Now Let $x'_5 = x'_5 + 7$, $(x'_1, x'_2, x'_3, x'_4, x'_5) = (1, 1, 6, 0, 6)$. Let $x'_5 = x'_5 + 7$ and $x'_4 = x'_4 + 7$, $(x'_1, x'_2, x'_3, x'_4, x'_5) = (1, 1, 7, 0, 5)$. Let $x'_5 = x'_5 + 7$ and $x'_4 = x'_4 + 7$ again, then $(x'_1, x'_2, x'_3, x'_4, x'_5) = (1, 2, 0, 0, 4)$. This means that we added $(n - x_j + 1) = (7 - 6 + 1) = 2$ times ω to η.

Although fast algorithm $A.3.3$ is much faster than the basic one if n is big, it still spends lots of time to do the non-use circulations. Therefore it is not the most effective algorithm for finding x_i's. To our surprise from the proof of the fast algorithm A.3.3, we can give the the fastest algorithm.

Fastest algorithm A.3.4: Let $x_1, x_2, ..., x_k$ be the bits of one of the ordered k digit $(n+1)$-nary numbers in increasing order of magnitude whose summation of bits are n. Let $\eta = x_1(n+1)^{k-1} + ... + x_k$ and let $x'_1, x'_2, ..., x'_k$ be the bits of the next number after η whose bits add to n. If $x_k > 0$, then $x'_k = x_k - 1, x'_{k-1} = x_{k-1} + 1$. If $\exists j > 1$ such that $x_{j+1} = x_{j+2} = x_k = 0$, then $x'_{j-1} = x_{j-1} + 1, x'_k = x_j - 1, x'_j = 0$. The remaining values don't change. In each step we will find the next $(n+1)$-ary number whose bits $x'_1, x'_2, ..., x'_k$ add to n.

Proof. From the proof of the fast algorithm $A.3.3$, we know that if $\exists j, x_{j+1} = x_{j+2} = x_k = 0$, then x_j must be carried into x_{j-1}. Then $x'_{j-1} = x_{j-1} + 1, x'_j = 0$. In order to satisfy $x'_1 + x'_2 + ... + x'_k = n$, x_k should be $x_j - 1$. And since $x_j > 0$, so $x_k = x_j - 1 \geq 0$, so at this time x_{j-1} need not to be carried into x_{j-2}. And each time, we find the exact x'_is satisfying $x'_1 + x'_2 + ... + x'_k = n$, so it is the fastest algorithm.

Note: This algorithm start with $x_1 = 0, x_2 = 0, ..., x_k = n$.

For example, suppose that $k = 6$ and $n = 9$. Let $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 3, 5, 0, 0, 0)$. In this case we have that $j = 3$, $x_j = 5$, and $x_{j-1} = 3$. Then we immediately get $(x'_1, x'_2, x'_3, x'_4, x'_5, x'_6) = (1, 4, 0, 0, 0, 4)$ by the fastest algorithm A.3.4.

Theorem A.3.5: Suppose $x_1+x_2+...+x_k = n$. Let $s_2 = x_2+x_3+...+x_k$, $s_3 = x_3+x_4+...+x_k$, ..., $s_{k-1} = x_{k-1} + x_k$, $s_k = x_k$. Then $0 \leq x_k \leq n$, $0 \leq x_{k-1} \leq n - s_k$, ..., $0 \leq x_3 \leq n - s_4$, $0 \leq x_2 \leq n - s_3$, $x_1 = n - s_2$.

Proof. Since $x_1+x_2+...+x_k = n$, we have that $x_k = n-x_1-x_2-...-x_{k-1} \leq n$. Therefore $0 \leq x_k \leq n$. Since $x_{k-1} + x_k \leq x_1 + x_2 + ... + x_k = n$, we have that $x_{k-1} \leq n - x_k = n - s_k$. Similarly, we can prove that $0 \leq x_3 \leq n - s_4$, $0 \leq x_2 \leq n - s_3$. Since $x_1 + x_2 + ... + x_k = n$, then $x_1 + s_2 = n$, and therefore $x_1 = n - s_2$.

Fastest algorithm A.3.6: Suppose $x_1, x_2, ..., x_k$ are bits of one of the ordered k digit $(n+1)$-nary numbers in increasing order of magnitude whose summation of bits are n, i.e., $x_1 + x_2 + ...+x_k = n$. Let $\eta = x_k(n+1)^{k-1}+x_{k-1}(n+1)^{k-2}+...+x_1$. Let the bits of the next $(n+1)$-nary numbers whose bits add to n be $x'_1, x'_2, ..., x'_k$. Let $s_2 = x_2+x_3+...+x_k$, $s_3 = x_3+x_4+...+x_k$, ..., $s_{k-1} = x_{k-1}+x_k$, and $s_k = x_k$. Consider the largest $i \in \{2, 3, ..., k-1\}$ such that $x_i = n - s_{i+1}$. Then set $x'_2 = ... = x'_i = 0$, $x'_{i+1} = x_{i+1} + 1$, $x'_{i+2} = x_{i+2}, ..., x'_k = x_k$, $x'_1 = n - s'_2$, where $s'_2 = x'_2+...+x'_k$. If there isn't such an i, then let $x'_k = x_k, x'_{k-1} = x_{k-1}, ..., x'_3 = x_3$, $x'_2 = x_2+1$, and $x'_1 = n - s'_2$. This process continues until $x'_k > n$.

Proof. The sequences of $x'_1, x'_2, ..., x'_k$ are bits of ordered continuous $(n+1)$-nary numbers. It is clear that $x'_1, x'_2, ..., x'_k$ satisfy theorem $A.3.5$ and $x'_1 + x'_2 + ... + x'_k = n$.

For example, suppose that $k = 5$, $n = 9$, and $(x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, 4, 2)$. $(s_3, s_4, s_5) = (7, 6, 2)$. $(n-s_3, n-s_4, n-s_5) = (2, 3, 7)$. Since $x_2 = 1 < 2 = n-s_3$, we set $x'_2 = x_2+1 = 2$, and $x'_1 = 9 - 9 = 0$. Therefore $(x'_1, x'_2, x'_3, x'_4, x'_5) = (0, 2, 1, 4, 2)$. The other steps are summarized as follows:

Table $A.3.1$ The procedures of Algorithm $A.3.6$

x'_1	x'_2	x'_3	x'_4	x'_5	s_3	s_4	s_5	$n-s_3$	$n-s_4$	$n-s_5$	$x'_2=n-s_3$
0	2	1	4	2	7	6	2	2	3	7	T
1	0	2	4	2	8	6	2	1	3	7	F
0	1	2	4	2	8	6	2	1	3	7	T
0	0	3	4	2	9	6	2	0	3	7	T
2	0	0	5	2	7	7	2	2	2	7	F
1	1	0	5	2	7	7	2	2	2	7	F
0	2	0	5	2	7	7	2	2	2	7	T
1	0	1	5	2	8	7	2	1	2	7	F
0	1	1	5	2	8	7	2	1	2	7	T
0	0	2	5	2	9	7	2	0	2	7	T
1	0	0	6	2	8	8	2	1	1	7	F
0	1	0	6	2	8	8	2	1	1	7	T
0	0	1	6	2	9	8	2	0	1	7	T
0	0	0	7	2	9	9	2	0	0	7	T
6	0	0	0	3	3	3	3	6	6	6	F
...		...									

$x'_3=n-s_4$	$x'_4=n-s_5$
F	F
F	F
F	F
T	F
F	F
F	F
F	F
F	F
F	F
T	F
F	F
F	F
T	F
T	T
F	F

In order to calculate $s_2, s_3, ..., s_k$ quickly, let $s_k = x_k$, $s_{k-1} = x_{k-1} + s_k$, ..., $s_2 = x_2 + s_3$. Although algorithm $A.3.6$ is a theoretically fast algorithm since each step produces an $(n+1)$-

nary number whose bits add to n, it takes time to calculate $s_2, ..., s_k$. Therefore it is slower than algorithm $A.3.4$. So algorithm $A.3.4$ is the fastest algorithm practically.

References

Alam, K. (1971). On selecting the most probable category. *Technometrics* **13** 843–850.

Cacoullos, T. and M. Sobel (1966). An inverse sampling procedure for selecting the most probable event in a multinomial distribution. In *Multivariate Analysis (P. R. Krishnaiah,1966)*, pp. 423–455. New York: Academic Press.

Casella, G. and R. L. Berger (2002). *Statistical Inference*. Second edition. Wadsworth Group, CA.

Gupta, S. S. and K. Nagel (1967). On selection and ranking procedures and order statistics from the multinomial distribution. *Sankhya B* **29** 1–34.

Hogg, R. V. and A. T. Craig (1995). *Introduction to Mathematical Statistics*. Fifth edition. Macmillan, New York.

Johnson, N. L. and D. H. Young (1960). Some applications of two approximations to the multinomial distribution. *Biometrika* **47** 463–469.

Panchapakesan, S., A. Childs, B. H. Humphrey, and N. Balakrishnan (1998). Inverse sampling procedures to test for homogeneity in a multinomial distribution. In *Order Statistics: Applications (N. Balakrishnan and C. R. Rao,1998)*, pp. 401–427. Elsevier Science B. V.

Patnaik, P. B. (1949). The non-central χ^2- and F-distributions and their applications. *Biometrika* **36** 202–232.

Ramey, J. T. and K. Alam (1979). A sequential procedure for selecting the most probable multinomial event. *Biometrika* **66** 171–173.

Rao, C. R. (1973). *Linear Statistical Inference and its Applications*. Second edition. John Wiley and Sons, New York.

Young, D. H. (1962). Two alternatives to the standard χ^2 test of the hypothesis of equal cell frequencies. *Biometrika* **49** 107–116.