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INFERENCE FOR ONE-SHOT DEVICE TESTING DATA

By

MAN HO LING, B.Sc., M.Phil.

A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

McMaster University

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TITLE: INFERENCE FOR ONE-SHOT DEVICE TESTING DATA

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Abstract

In this thesis, inferential methods for one-shot device testing data from accelerated life-test are developed. Due to constraints on time and budget, accelerated life-tests are commonly used to induce more failures within a reasonable amount of test-time for obtaining more lifetime information that will be especially useful in reliability analysis. One-shot devices, which can be used only once as they get destroyed immediately after testing, yield observations only on their condition and not on their real lifetimes. So, only binary response data are observed from an one-shot device testing experiment. Since no failure times of units are observed, we use the EM algorithm for determining the maximum likelihood estimates of the model parameters. Also, inference for the reliability at a mission time and the mean lifetime at normal operating conditions are also developed.

The thesis proceeds as follows. Chapter 2 considers the exponential distribution with single-stress relationship and develops inferential methods for the model parameters, the reliability and the mean lifetime. The results obtained by the EM algorithm are compared with those obtained from the Bayesian approach. A one-shot device testing data is analyzed by the proposed method and presented as an illustrative example. Next, in Chapter 3, the exponential distribution with multiple-stress relationship is considered and corresponding inferential results are developed. Jackknife technique is described for the bias reduction in the developed estimates. Interval estimation for the reliability and the mean lifetime are also discussed based on observed information matrix, jackknife technique, parametric bootstrap method, and transformation technique. Again, we present an example to illustrate all the inferential methods developed in this chapter. Chapter 4 considers the point and interval estimation for the one-shot device testing data under the Weibull distribution with multiple-stress relationship and illustrates the application of the proposed methods in a study involving the development of tumors in mice with respect to risk factors such as sex, strain of offspring, and dose effects of benzidine dihydrochloride. A Monte Carlo simulation study is also carried out to evaluate the performance of the EM estimates for different levels of reliability and different sample sizes. Chapter 5 describes a general algorithm for the determination of the optimal design of an accelerated life-test plan for one-shot device testing experiment. It is based on the asymptotic variance of the estimated reliability at a specific mission time. A numerical example is presented to illustrate the application of the algorithm. Finally, Chapter 6 presents some concluding remarks and some additional research problems that would be of interest for further study.

KEY WORDS: One-shot device testing, survival/sacrifice data, carcinogenic data, binary data, censoring, accelerated factor, log-linear link function, exponential distribution, Weibull distribution, EM algorithm, Bayesian method, asymptotic method, parametric bootstrap, jackknife, least-squares method, point estimation, confidence intervals, transformation approach, optimal test plan.

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List of Publications from the Thesis

- Balakrishnan, N. and Ling, M.H. "EM algorithm for one-shot devices testing under the exponential distribution. Computational Statistics and Data Analysis" (56), pp 502-509, 2012. (Chapter 2)
- Balakrishnan, N. and Ling, M.H. "EM algorithm for inference on multiplestress model for one-shot device testing data under exponential distribution" IEEE transactions on reliability. Accepted. (Chapter 3)
- 3. Balakrishnan, N. and Ling, M.H. "EM algorithm for one-shot device accelerated life-test data based on Weibull lifetime distribution with scale and shape parameters varying over stress" Submitted to IEEE transactions on reliability. (Chapter 4)
- Balakrishnan, N. and Ling, M.H. "Optimal accelerated life-test plans for oneshot device testing" Submitted to IEEE transactions on reliability. (Chapter 5)

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Chapter 1

Introduction

1.1 Motivation

The study of one-shot device from accelerated life-test (ALT) data is motivated by the work of Fan *et al.* [19]. They developed the Bayesian approach for one-shot device testing along with an accelerating factor. In spite of small sample sizes in their simulation study, their Bayesian estimator incorporating normal prior yields accurate inference on the model parameters, the reliability as well as the mean lifetime under normal operating conditions. In their development, a strong assumption that prior information on success rate is very reliable was made, which means that the reliability estimates attained are based on the information that it is close to the true value. For this purpose, in their simulation study, the prior information was generated from a distribution which is around the true value and with a small variability. Unfortunately, a reliable prior belief may not always be available in practice and thus it becomes somewhat subjective. Furthermore, Dunson and Dinse [14] pointed out that some parameter estimates are sensitive to the selection of prior in case of small samples. Fan *et al.* [19] also made a similar remark that the results were dominated by the prior distribution in the case of zero-failure. We, therefore, adopt here the conventional method of analysis for such data by finding maximum likelihood estimates (MLEs) through the EM algorithm. In problems involving one-shot device testing data, the reliability of devices and the mean lifetime at normal operating conditions are often of primary interest rather than the model parameters themselves, and so we address the corresponding estimation problems as well.

1.2 Accelerated Life-Test

Accelerated life-tests play an important role in engineering. Due to intense global competition and high customer expectations, manufacturers are under pressure to produce products of high reliability and with longer life. In such a situation, under a conventional life-test, there will be very few failures or even no failure within a limited time under normal operating conditions. Thus, collection of sufficient lifetime information of such highly reliable products becomes extremely difficult. Therefore, it becomes necessary to draw inference about the relationship between the lifetime of products and external stress factors. Consequenctly, accelerated life-tests (ALTs) are commonly used by adjusting controllable factors in order to induce rapid failures at high stress levels. This would result in reducing experimental time as well as the cost of conducting the experiment, and at the same time enable to collect more lifetime information within a limited period of time. Temperature, humidity, pressure and voltage are often used as accelerating factors.

ALTs are popular in engineering practice for many products and materials. Trevisanello et al. [58] reported an ALT on high brightness light emitting diodes by submitting the devices to high temperature and high direct current levels. During aging, degradation mechanism like light output decay was detected. They used the time to reach the 70% of light output as the lifetime of the device and observed a nearly exponential decay kinetic. Recently, Zhang and Ba [71] carried out an ALT by increasing direct current levels to predict the service life of concrete in a chloride environment. The time that elapsed before critical chloride concentration leading to reinforcement corrosion is defined as the lifetime of concrete in their study. In addition, Meeker et al. [38] studied accelerated degradation data on integrated-circuit (IC) devices and used the accelerated degradation model to develop inference and prediction about its lifetime distribution at standard operating temperature. In their work, they have also mentioned many applications of ALTs and described methods for analyzing such data. For statistical purposes, Lu et al. [35] compared degradation analysis and traditional failure time analysis in terms of asymptotic variances of estimators of a quantile of the lifetime distribution. Rodrigues et al. [53] presented two approaches based on the likelihood ratio statistic and the posterior Bayes factor for comparing several exponential

accelerated life models.

Since it is necessary to extrapolate the failure data collected from an ALT from high stress levels to normal operating conditions, a suitable model motivated by physical justification is required, relating the lifetime of products to stress factors, to draw inference on such characteristics as the reliability and the mean lifetime of devices at normal operating conditions. In this regard, Nelson [43] classified ALTs into three different types, which are as follows:

1. Constant-stress test

Each test item is subjected to a constant stress level, but stress level may differ from item to item.

2. Step-stress test

Each test item undergoes a pattern of increasing stress levels for fixed period of time. A test item initially starts at a pre-specified constant stress level for a specified length of time. The stress levels are then increased and maintained step by step at some pre-specified points of time during the experiment.

3. Progressive-stress test

Each test item is subjected to continuously increasing stress levels over time.

Nelson [43] also mentioned that the step-stress and progressive-stress tests are advanced techniques in ALTs. These approaches precipitate failures more quickly for analysis. However, there is a principal disadvantage in these tests since the assumed model must take properly into account the cumulative effect of exposure at successive stress levels and that it must provide an estimate of lifetime under constant stress. Since, in practice, it is very difficult to properly model the acceleration, it becomes difficult to predict the lifetime under normal operating conditions based on observations from accelerated levels. Relatively, constantstress test is easiest to conduct as well as to model for reliability estimation, and hence is preferable over the step-stress and progressive-stress tests. We, therefore, focus our attention here on the constant-stress test in this thesis.

1.3 Life-Stress Relationships

In ALTs, failure rate is required to relate to stress factors such that measurements taken during the experiment can then be extrapolated back to the expected performance under normal operating conditions. A curve representing the relationship between the failure rate and the stress factors is needed and a simple model such as linear model may not suffice to describe the lifetime of products against the stress factors. Wang and Kececioglu [64] and Pascual [50] presented three common relationships between lifetime of products and stress factors, which are as follows:

1. Log-linear relationship

It is commonly used in practice due to its mathematical convenience. It shows the relative importance of stress factors in influencing the failure behavior, regardless of whether the model is correct or not. The parameter $\theta(x)$ relates to a stress factor x in this case in the form

$$\theta(x) = \exp(\gamma_0 + \gamma_1 x); \tag{1.1}$$

2. Inverse power law relationship

It is useful for describing the lifetime as a function of applied voltage. The parameter $\theta(V)$ relates to applied voltage V in the form

$$\theta(V) = \frac{1}{\gamma_0 V^{\gamma_1}};\tag{1.2}$$

3. Arrhenius accelerated relationship

Based on the Arrhenius Law for simple chemical-reaction rates, it is one of the commonly used acceleration models that predicts the failure time varying with temperature. The parameter $\theta(T)$ relates to temperature T in degree Celsius as follows:

$$\theta(T) = \exp\left(\gamma_0 + \gamma_1 \frac{11605}{T + 273.15}\right).$$
 (1.3)

In fact, lifetime distribution models with the log-linear relationships with covariates are also often used in survival analysis.

1.4 Types of Censoring

For lifetime data analysis, a complete lifetime data in which all failure times are collected from the experiment is the most preferable one as it would result in most precise analysis. In reality, however, there will be difficulties in observing failure times of all units under test. Such incomplete data frequently arise in life-tests and are referred to as censored data. Depending on the nature and form of the life-testing experiment, different forms of censoring may occur in the data.

1.4.1 Right Censoring

Due to constraints on time and cost, right censoring is commonly encountered in life-testing experiments. Under this form of censoring, while some lifetimes will be completely observed, others will be known only to be beyond some times. There are two main types of right censoring, which are as follows:

1. Type-I (Time-censoring)

The life-test is terminated at a pre-fixed time, resulting in a fixed censoring time and a random number of failures during the experimental period;

2. Type-II (Failure-censoring)

The life-test is terminated as soon as a pre-fixed number of failures have been observed, resulting in a random censoring time and a fixed number of failures during the experimental period.

Type-I censoring is quite practical as it restricts the duration of test to a preset termination time. It, however, results in a random number of failures and consequently may end up with ineffective inference due to the realization of very few failures. In contrast, the number of failures is guaranteed (at a pre-fixed number) under Type-II censoring and therefore assists in the planning adequate tests. However, the random termination time poses difficulties from managerial point of view since the duration of the test is not known in advance in this case.

Since it is common to face right censored data in practice, many survival models based on right censored data have been studied extensively in the literature. To mention a few, for example, Dupuy [15] proposed likelihood ratio type tests to detect change in a hazard regression model based on right-censored data. Emura *et al.* [18] developed a goodness-of-fit test for Archimedean copula models based on right censored sample. Zhao and Zhao [72] considered two samples in the presence of right censoring and constructed confidence intervals for the ratio or difference of two hazard functions by using smoothed empirical likelihood methods. Lin *et al.* [32] presented a method for testing goodness-of-fit based on samples with Type-II right censoring. Joarder *et al.* [25] studied statistical inferential methods for Weibull distribution based on Type-I right censored samples.

1.4.2 Interval Censoring

In some situations, test items are inspected for failure at many time points and one only knows that items failed in some intervals between two contiguous inspections. Such lifetime data are said to be interval censored and arise naturally when the test items are not constantly monitored. More specifically, if an item is inspected at 100 hours and is still operating and then inspected at 200 hours but is no longer operating, we only know that the failure occurred between 100 and 200 hours. In survival analysis, interval censoring occurs when the response times arise from a clinical trial or a longitudinal study in which there is a periodic followup. Yu [69] proposed Bayesian MCMC method to analyze such interval censored data from an AIDS cohort study and a population-based dementia study. Chen *et al.* [10] presented a full likelihood approach based on the proportional hazards frailty model for a bivariate current status data arising from a tumorigenicity experiment. Finkelstein [20] studied proportional hazards regression model for animal tumorigenicity study and breast cosmesis study when the available data are interval censored.

1.5 One-Shot Device Testing Data

In this thesis, one-shot device testing data, which is an extreme case of interval censoring, is studied. Since one-shot devices can be used only once and are destroyed immediately after use, one can only know whether the failure time is either before or after a specific time. Since the condition of a test device at a specific inspection time is observed rather than its actual lifetime, binary data is observed naturally from such experiments. They are also known as current status data in survival analysis. As a typical example, Fan *et al.* [19] studied electro-explosive devices and, in particular, developed inferential methods for their reliability. The devices are detonated^{*} by inducing a current to excite the inner powder, and they

^{*}In the case of electro-explosive devices, the success or failure is based on whether it was successfully detonated or not; in biological systems, however, it is based on whether the system died (sacrificed) or not, but in this thesis, we refer to this in all cases by 'detonation'.

can not be re-used after detonation. No matter whether the detonation is successful or not, the lifetime of devices can never be recorded. The lifetimes are either left- or right-censored, with the lifetime being less than the inspection time if the test outcome is a failure (resulting in left censoring) and the lifetime being more than the inspection time if the test outcome is a success (resulting in right censoring). Yet another example of this type is in the study of reliability of bombs and missiles faced by Australian Defence Organization. In this situation, Yates and Mosleh [68] developed a Bayesian approach for the reliability estimation. One-shot devices such as fire extinguishers and munitions have also been discussed for maintenance and monitoring by Newby [46]. Morris [40] analyzed a battery data from destructive life-tests in which the batteries were stored under a mildly accelerated aging temperature and a relatively light load. In addition to these situations, one-shot device data also arise naturally in tests of space shuttles, military weapons and automobile air bags. Note that in all these scenarios, tested devices can not be used any further since the test results in their destruction.

Suppose devices are placed in I testing conditions, wherein, for i = 1, 2, ..., I, K_i devices are subjected to J independent stress factors, $\{x_{ij}, j = 1, 2, ..., J\}$, and under inspection at time IT_i . In the *i*-th testing condition, number of failures, n_i , is observed. We assume that the lifetime of the units, $\{t_{ik}, i = 1, 2, ..., I, k = 1, 2, ..., I, k = 1, 2, ..., K_i\}$, has a distribution with probability density function, $f(t; \theta_i)$, and cumulative distribution function, $F(t; \theta_i)$, where the parameter, θ_i , is assumed to

Table 1.1: Data on one-shot device testing at various stress levels collected at different inspection times.

Testing	Inspection	Numbers of	Numbers of		Covariates	
condition	time	devices	failures	Stress 1		Stress J
1	IT_1	K_1	n_1	<i>x</i> ₁₁		x_{1J}
2	IT_2	K_2	n_2	x_{21}		x_{2J}
÷	:	÷	÷	÷		÷
Ι	ITI	K_I	n_I	x_{I1}		x_{IJ}

relate to the stress levels in a log-linear form as

$$\theta_i = \exp\left(\sum_{j=0}^J a_j x_{ij}\right). \tag{1.4}$$

Note that $x_{i0} = 1$ for all *i*, corresponding to constant effect on the parameter in the model. The data thus observed can be summarized as in Table 1.1.

For ALT, it is reasonable to make an assumption that $a_j > 0$, since the accelerating factor are supposed to increase the failure rate thus decreasing the failure time. But in biomedical studies, it is not necessary to have this restriction since stress factors such as an increase in the dosage of medicine may sometimes help prolong the survival time. So, we consider here a general situation that allows a_j to be negative as well.

For notational convenience, we denote $\mathbf{z} = \{IT_i, K_i, n_i, x_{ij}, i = 1, 2, ..., I, j = 1, 2, ..., J\}$ for the observed data and $\boldsymbol{\theta} = \{\theta_i, i = 1, 2, ..., I\}$. Then, the likelihood

function based on this observed data is given by

$$L(\boldsymbol{\theta}; \mathbf{z}) \propto \prod_{i=1}^{I} \left[F(IT_i; \theta_i) \right]^{n_i} \left[1 - F(IT_i; \theta_i) \right]^{K_i - n_i}.$$
 (1.5)

1.6 EM Algorithm

Maximum likelihood method is commonly adopted in analyzing reliability data due to its well-known optimality properties. The EM algorithm is a quite an useful and powerful technique for numerically determining the MLEs in the presence of missing data, even though the final estimates of the parameters are not in a closed-form; see McLachlan and Krishnan [36] and Casella and Berger [8] for an overview of this technique. The EM algorithm proceeds by alternating between the expectation step (E-step) and the maximization step (M-step) in each iteration. In the E-step, the expected log-likelihood of the complete data, conditional on the observed data and the current parameters, is computed. In the M-step, updated estimates of the parameters are computed by maximizing the expected log-likelihood function. This process is repeated until convergence occurs to a desired level of accuracy. Thus, two different likelihood problems are considered in the EM algorithm, one involving an approximation of the missing data and another involving the maximization of the corresponding likelihood function. The problem that we are focusing here, namely, in analyzing an one-shot device testing data, is an incomplete data problem, and so we solve it by first approximating the

missing data and then using them to update the estimate of the parameter as the solution to the complete data problem.

Given a likelihood function $L_c(\boldsymbol{\theta}; x, z)$, where x is the observed data and z represents the unobserved latent data, the MLEs are determined by the marginal likelihood of the observed data $L_c(\boldsymbol{\theta}; x)$. The EM algorithm seeks to find the MLEs by iteratively applying the following two steps:

- 1. Calculate the expected value of the log-likelihood function, with respect to the conditional distribution of z, given x, under the current estimate of the parameter $\boldsymbol{\theta}^{(m)}$;
- 2. Find the next iterate of the parameter $\boldsymbol{\theta}^{(m+1)}$ by maximizing the quantity

$$E[log(L_c(\boldsymbol{\theta}; x, z) | \boldsymbol{\theta}^{(m)}, x)]$$

Much work has been done on the estimation of parameters in incomplete data problems through the EM algorithm. For instance, Ng *et al.* [47] developed inference for the lognomal and Weibull distributions based on progressively censored data. Kundu and Dey [30] and Nandi and Dewan [42] considered the Marshall-Olkin bivariate Weibull distribution under random censoring using the EM algorithm and analyzed a soccer data from UEFA Champions League. Chen *et al.* [10] also presented the EM algorithm based on a proportional hazards frailty model for analyzing a bivariate current status data arising from a tumorigenicity experiment. Scheike and Sun [54] presented the EM algorithm for the Cox proportional regression model for right censored survival data. Chen and Lio [9] examined the EM algorithm for generalized exponential distribution under progressively Type-I interval censored data. Furthermore, the EM algorithm has been applied in a wide variety of problems involving missing values; for example, in studies of competing risks model [11], astroparticle physics studies [1], marine studies [41], and studies of genetics, genomics and public health [31]. As mentioned before, under one-shot device testing, one can observe only the condition of the devices at inspection times rather than the real lifetimes. Thus, the lifetimes of the one-shot devices are censored, in fact, on both sides, and consequently the EM algorithm becomes quite appropriate for determining the MLEs of the model parameters, and to develop subsequent likelihood inferential methods.

1.7 Scope of the Thesis

The rest of this thesis is organized as follows. In Chapters 2, 3 and 4, the EM algorithm is proposed for different lifetime models. The E-step and the M-step are developed for the problem under different considerations. Chapter 2 considers the exponential distribution with single-stress relationship. Estimation of the model parameters, the reliability, and the mean lifetime under a Bayesian approach are also described and compared with those obtained from the EM algorithm. A one-shot device testing data is analyzed to illustrate the proposed methods. Next, in Chapter 3, the exponential distribution with multiple-stress relationship is consid-

ered. Jackknife technique is described for bias reduction of the proposed estimates. In addition to point estimation, confidence intervals for the reliability and the mean lifetime are discussed based on four methods of variance estimation. Here again, we present an example to illustrate all the inferential methods developed in this chapter. Chapter 4 considers the point and interval estimation based on one-shot device testing data under the Weibull distribution with multiple-stress relationship and illustrates the proposed methods by using the data from a study of tumors in mice induced by benzidine dihydrochloride. Simulation study is also carried out to show the performance of the EM estimates for different levels of reliability and different sample sizes in each chapter. Chapter 5 describes an algorithm for the determination of the optimal accelerated life-test plan with an example. Chapter 6 finally provides some concluding remarks and also points out some further problems of interest.
Chapter 2

Exponential Lifetime Distribution with Single-Stress Model

2.1 Introduction

A number of Bayesian models for one-shot device testing data have been studied. Cai *et al.* [7] suggested a Bayesian proportional hazards model for analyzing an uterine fibroid data from an epidemiological study. Fan *et al.* [19] developed the Bayesian approach for one-shot device testing along with an accelerating factor of temperature, in which the failure time of the devices is assumed to follow an exponential distribution. As mentioned before, the prior information on the success rate is assumed to be very reliable in their development. That is, the reliability estimates attained are based on the information that it is close to the true value. In addition, the prior information was generated from a distribution which is around the true value and with a small variability in their simulation study. For these reasons, their Bayesian estimates incorporating normal prior yields precise inference on the model parameters, the reliability as well as the mean lifetime under normal temperature, even when the sample sizes are small.

In this Chapter, the EM algorithm is, therefore, developed for finding the MLEs of the model parameters, the reliability and the mean lifetime of devices under normal temperature, and this method is then compared with the above mentioned Bayesian approach. We show that the proposed method yields quite reliable and efficient estimates, and demonstrate that the EM algorithm is quite useful for analyzing such one-shot device testing data. Here, equal number of devices is assumed at each testing condition, that is, $K_1 = K_2 = \cdots = K_I = K$.

2.2 Model Description

Consider a reliability testing experiment in which K devices are placed under temperature x_i and tested at time IT_i , where i = 1, 2, ..., I. It is worth noting that a successful detonation occurs if its lifetime is beyond the inspection time, whereas the lifetime will be before the inspection time if the detonation is a failure. For each temperature x_i , the number of failures n_i is then recorded at each inspection time IT_i . We assume here that the true lifetimes t_{ik} , where i = 1, 2, ..., I, k = 1, 2, ..., K, are independent and identically distributed exponential random variables with probability density function (pdf) and cumulative distribution function (cdf) as

$$f(t) = \lambda e^{-\lambda t}, \qquad t \ge 0, \tag{2.1}$$

and

$$F(t) = 1 - e^{-\lambda t}, \qquad t \ge 0,$$
 (2.2)

where $\lambda > 0$ is the failure rate, respectively. Here, we relate the parameter λ to an accelerating factor of temperature x_i through a log-linear link function of the form

$$\lambda_i = \alpha_0 e^{\alpha_1 x_i}, \qquad x_i > 0. \tag{2.3}$$

In the analysis of lifetime data, the hazard function h(t) plays an important role and it is the instantaneous rate of failure time t, meaning the failure rate at time t conditional on survival until time t or later, i.e.,

$$h(t)\delta t = \Pr(t < T < t + \delta t | T > t) = \frac{f(t)\delta t}{R(t)},$$
(2.4)

where R(t) = 1 - F(t). It follows that the hazard rate, under exponential distribution, is then simply

$$h(t) = \frac{f(t)}{R(t)} = \lambda.$$
(2.5)

Figure 2.1 presents a plot of the pdf, the reliability function (R(t) = 1 - F(t)), and the hazard function over time t for some choices of the failure rate λ . The hazard function is constant over time, which means that the instantaneous failure rate never changes. Moreover, the reliability drops rapidly as λ becomes larger. The exponential distribution is the simplest model for analyzing lifetime data, and

Figure 2.1: Plots of the exponential pdf, the reliability, and the hazard function for different choices of failure rate.



it possesses an important memoryless property, that is,

$$\Pr(T > t + s | T > s) = \Pr(T > t), \quad \text{for all } s, t \ge 0.$$

The reliability function R(t) at time t and the mean lifetime E(T) under normal temperature x_0 are given by

$$R(t) = e^{-\lambda_0 t} = \exp(-\alpha_0 e^{\alpha_1 x_0} t), \qquad t \ge 0,$$
(2.6)

and

$$E(T) = \frac{1}{\lambda_0} = \frac{1}{\alpha_0 e^{\alpha_1 x_0}}.$$
 (2.7)

2.3 Point Estimation of Parameters of Interest

2.3.1 EM Algorithm based on an Iterative Formula

In the EM algorithm, the parameters will be estimated in the M-step while the expected lifetimes will be approximated in the E-step. The failure rate is related to an accelerating factor of temperature through a log-linear link function in Eq. (2.3). The log-likelihood function of α_0 and α_1 based on the complete data is given by

$$\ell_{c}(\alpha_{0}, \alpha_{1}) = \sum_{i=1}^{I} \sum_{k=1}^{K} (\log \alpha_{0} + \alpha_{1}x_{i} - \alpha_{0}e^{\alpha_{1}x_{i}}t_{ik})$$
$$= IK \log \alpha_{0} + \alpha_{1}KX - \alpha_{0} \sum_{i=1}^{I} (e^{\alpha_{1}x_{i}}T_{i}^{*}), \qquad (2.8)$$

where $X = \sum_{i=1}^{I} x_i$ and $T_i^* = \sum_{k=1}^{K} t_{ik}$.

In the maximization step, upon setting $\frac{\partial \ell_c(\alpha_0, \alpha_1)}{\partial \alpha_0} = 0$ and $\frac{\partial \ell_c(\alpha_0, \alpha_1)}{\partial \alpha_1} = 0$,

we obtain the estimates of the parameters α_0 and α_1 as solutions of the equations

$$\alpha_0 = \frac{IK}{\sum_{i=1}^{I} (e^{\alpha_1 x_i} T_i^*)}$$
(2.9)

and

$$\frac{X}{I} = \frac{\sum_{i=1}^{I} (x_i T_i^* e^{\alpha_1 x_i})}{\sum_{i=1}^{I} (T_i^* e^{\alpha_1 x_i})}.$$
(2.10)

For the solution of α_1 from Eq. (2.10), we adopt the iterative formula

$$\alpha_1^{(l+1)} = x_I^{-1} \log \left\{ \frac{\sum_{i=1}^{I-1} \left[(X - Ix_i) T_i^* e^{\alpha_1^{(l)} x_i} \right]}{(Ix_I - X) T_I^*} \right\}.$$
 (2.11)

In the expectation step, the mean lifetime $E[T_i^*|\lambda_i, \mathbf{z}]$ under the temperature x_i (needed for solving the above equations) can be obtained as follows:

$$E[T_i^*|\lambda_i, \mathbf{z}] = \sum_{k=1}^{K} E[T_{ik}|\lambda_i, \mathbf{z}]$$
$$= \left(\frac{n_i \int_0^{IT_i} t\lambda_i e^{-\lambda_i t} dt}{\int_0^{IT_i} \lambda_i e^{-\lambda_i t} dt} + \frac{(K - n_i) \int_{IT_i}^{\infty} t\lambda_i e^{-\lambda_i t} dt}{\int_{IT_i}^{\infty} \lambda_i e^{-\lambda_i t} dt}\right)$$
$$= \left[n_i \left(\frac{1}{\lambda_i} - \frac{(IT_i)e^{-\lambda_i(IT_i)}}{1 - e^{-\lambda_i(IT_i)}}\right) + (K - n_i) \left(IT_i + \frac{1}{\lambda_i}\right)\right]. \quad (2.12)$$

In the above expression, the first term accounts for all devices that failed in the test resulting in lifetimes that are before the inspection times, which corresponds to left censoring. On the other hand, the second term accounts for all devices that detonated successfully resulting in lifetimes that are beyond the inspection times, which corresponds to right censoring.

Suppose the estimates of the parameters α_0 and α_1 at the *m*-th step are $\alpha_0^{(m)}$ and $\alpha_1^{(m)}$, respectively. Then, the (m + 1)-th step of the EM algorithm proceeds as follows:

- 1. Compute $E[T_i^*|\lambda_i, \mathbf{z}]$ in Eq. (2.12) by using $\alpha_0^{(m)}$ and $\alpha_1^{(m)}$, for $i = 1, 2, \ldots, I$;
- 2. Starting with $\alpha_1^{(m)}$, for fixed $E[T_i^*|\lambda_i, \mathbf{z}]$ (i = 1, 2, ..., I), find $\alpha_1^{(m+1)}$ iteratively from Eq. (2.11);
- 3. Given $\alpha_1^{(m+1)}$, compute $\alpha_0^{(m+1)}$ from Eq. (2.9);
- 4. Repeat Steps 1-3 by using $\alpha_0^{(m+1)}$ and $\alpha_1^{(m+1)}$, until convergence occurs to a desired level of accuracy.

Given the estimates of α_0 and α_1 , denoted by $\hat{\alpha}_0$ and $\hat{\alpha}_1$, due to the invariance property of the maximum likelihood estimates that the estimate of a function is simply the function evaluated at the MLEs of the model parameters, we can therefore plug-in those estimates into the corresponding functions to obtain inference on the reliability $\hat{R}(t)$ at time t and the mean lifetime $\widehat{E(T)}$ under normal temperature of 25°C as follows:

$$\hat{R}(t) = \exp(-\hat{\alpha}_0 e^{25\hat{\alpha}_1} t)$$
 (2.13)

and

$$\widehat{E(T)} = \frac{1}{\hat{\alpha}_0 e^{25\hat{\alpha}_1}}.$$
(2.14)

2.3.2 Bayesian Approach

In the Bayesian approach, the parameter itself is thought to be a random quantity whose variation can be described by a prior distribution. A sample is taken from the population and the distribution is updated with this sample. We describe here the inference on the reliability of the electro-explosive devices through the Bayesian approach, as developed by Fan *et al.* [19].

Let p_i denote the associated survival probability under temperature x_i at time IT_i . We then have

$$p_{i} = 1 - F(IT_{i}|\lambda_{i}) = \exp(-\lambda_{i}(IT_{i})) = \exp(-\alpha_{0}e^{\alpha_{1}x_{i}}(IT_{i})).$$
(2.15)

Now, given the data \mathbf{z} , the likelihood function of α_0 and α_1 is given by

$$L(\alpha_0, \alpha_1 | \mathbf{z}) = \prod_{i=1}^{I} p_i^{K-n_i} (1-p_i)^{n_i}$$

= $\prod_{i=1}^{I} \exp(-\alpha_0 (K-n_i) e^{\alpha_1 x_i} (IT_i)) \{1 - \exp(-\alpha_0 e^{\alpha_1 x_i} (IT_i))\}^{n_i}.$
(2.16)

Let $\pi(\alpha_0, \alpha_1)$ be the joint prior density of (α_0, α_1) . Then, given the data \mathbf{z} , the joint posterior density of (α_0, α_1) is given by

$$\pi(\alpha_0, \alpha_1 | \mathbf{z}) = \frac{L(\alpha_0, \alpha_1 | \mathbf{z}) \pi(\alpha_0, \alpha_1)}{\int \int L(\alpha_0, \alpha_1 | \mathbf{z}) \pi(\alpha_0, \alpha_1) d\alpha_0 d\alpha_1},$$
(2.17)

where $L(\alpha_0, \alpha_1 | \mathbf{z})$ is the likelihood function of α_0 and α_1 . Since the denominator in the above equation is not in closed-form, the Markov Chain Monte Carlo (MCMC) method is used to generate approximate posterior samples of (α_0, α_1) , say $(\alpha_0^{(r)}, \alpha_1^{(r)}), r = 1, 2, ..., R$, and the marginal sample means can then be used as approximate Bayesian point estimates of α_0 and α_1 , given by

$$\hat{\alpha}_0 = \frac{1}{R} \sum_{r=1}^R \alpha_0^{(r)} \tag{2.18}$$

and

$$\hat{\alpha}_1 = \frac{1}{R} \sum_{r=1}^R \alpha_1^{(r)}.$$
(2.19)

Then, the reliability at time t under normal temperature of 25°C can be readily estimated as

$$\hat{R}(t) = \frac{1}{R} \sum_{r=1}^{R} \exp(-\alpha_0^{(r)} e^{25\alpha_1^{(r)}} t).$$
(2.20)

In analogous manner, we can estimate the mean lifetime under normal temperature of 25° C by

$$\widehat{E(T)} = \frac{1}{R} \sum_{r=1}^{R} \frac{1}{\alpha_0^{(r)} e^{25\alpha_1^{(r)}}}.$$
(2.21)

In the choice of prior distribution, Fan *et al.* [19] suggested the use of normal prior in their Bayesian method. This prior information is appropriate to devices with moderate to high reliabilities. The joint prior density of (α_0, α_1) is given by

$$\pi(\alpha_0, \alpha_1 | \{x_i, IT_i, i = 1, 2, \dots, I\}) \propto \left\{ \sum_{i=1}^{I} \left[\exp(-\alpha_0 e^{\alpha_1 x_i} (IT_i)) - \hat{p}_i \right]^2 \right\}^{-I/2},$$
(2.22)

where \hat{p}_i represents the prior belief of the success rate p_i . Consequently, it results in the joint posterior density of (α_0, α_1) as

$$\pi(\alpha_0, \alpha_1 | \mathbf{z}) \propto \prod_{i=1}^{I} L(\alpha_0, \alpha_1 | \mathbf{z}) \left\{ \sum_{i=1}^{I} \left[\exp(-\alpha_0 e^{\alpha_1 x_i} (IT_i)) - \hat{p}_i \right]^2 \right\}^{-I/2}.$$
 (2.23)

It is worth noting here that the prior belief \hat{p}_i was supposed in Fan *et al.* [19] to be quite reliable to the true p_i , where \hat{p}_i are generated from a Beta (α_i^*, β_i^*) distribution with α_i^* and β_i^* being selected so that $E(\hat{p}_i) = p_i$ and $var(\hat{p}_i) = c^2$, where c^2 specifies the precision of the belief. Given c^2 , we then have

$$\alpha_i^* = p_i \left(\frac{p_i (1 - p_i)}{c^2} - 1 \right)$$
(2.24)

and

$$\beta_i^* = (1 - p_i) \left(\frac{p_i(1 - p_i)}{c^2} - 1 \right).$$
(2.25)

Here, we also propose another prior information for incorporating into the Bayesian approach. Since we know the number of failures at each time and each temperature, the prior belief can be viewed as

$$\tilde{p}_i = \frac{K - n_i}{K} = 1 - \frac{n_i}{K}.$$
(2.26)

Though the prior belief, \tilde{p}_i , will not be reliable in the case of small sample sizes, it would give us a better understanding of the importance of the prior belief to the Bayesian approach in the ensuing simulation study.

2.4 Illustrative Example

Fan *et al.* [19] presented a one-shot device testing data. There were 30 devices tested at temperatures $\{35, 45, 55\}$ each, of which 10 units were detonated at times $\{10, 20, 30\}$ each. The number of failures observed is summarized in Table 2.1. There were in all 48 failures out of a total of 90 devices that were tested in this one-shot device testing experiment. We now use these one-shot device testing data to illustrate the proposed EM algorithm method and the Bayesian approach

Testing	Inspection	Numbers of	Numbers of	Stress factor
condition	time	devices	failures	temperature (°C)
1	10	10	3	35
2	20	10	3	35
3	30	10	7	35
4	10	10	1	45
5	20	10	5	45
6	30	10	7	45
7	10	10	6	55
8	20	10	7	55
9	30	10	9	55

Table 2.1: The number of failures recorded under temperatures 35, 45, 55 (in °C) at inspection times 10, 20, 30, respectively, in a one-shot device testing.

Table 2.2: Point estimates for the model parameters, the reliabilities at time $t = \{10, 20, 30\}$, and the mean lifetime at normal temperature of 25°C for the data presented in Table 2.1.

	\hat{lpha}_0	\hat{lpha}_1	$\hat{R}(10)$	$\hat{R}(20)$	$\hat{R}(30)$	$\widehat{E(T)}$
EM algorithm	0.0049	0.0473	0.8530	0.7277	0.6208	62.9179
Bayesian with \tilde{p}_i	0.0084	0.0392	0.8236	0.6803	0.5635	55.3268

with normal prior and attainable prior belief \tilde{p}_{ij} , as discussed in Section 2. The estimates $\hat{\alpha}_0, \hat{\alpha}_1, \hat{R}(10), \hat{R}(20), \hat{R}(30)$ and $\widehat{E(T)}$ obtained by these two methods are all presented in Table 2.2.

Table 2.2 shows that the estimates of the model parameters by the EM algorithm and the Bayesian approach are slightly different, but the estimates of the reliabilities at mission times (time points in the future at which we are interested in the reliability of the unit) $t = \{10, 20, 30\}$ are close. Also, the mean lifetime obtained by the EM algorithm is greater than that obtained from the Bayesian approach.

2.5 Simulation Results

In this section, we investigate the performance of the EM algorithm and the Bayesian method incorporated with the prior beliefs \hat{p}_i generated from a beta distribution and \tilde{p}_i determined from the data [See Eq. (2.26)]. Under each temperature {35°C, 45°C, 55°C}, K devices are tested at each inspection time {10, 20, 30}. Note that the total number of devices tested is therefore $9 \times K$. All methods are examined under 12 conditions: choices of $(\alpha_0, \alpha_1) = (0.008, 0.05)$ for low reliability, $(\alpha_0, \alpha_1) = (0.004, 0.05)$ for moderate reliability, and $(\alpha_0, \alpha_1) = (0.001, 0.05)$ for high reliability when K = 10 (small sample size), K = 50 (moderate sample size), and K = 100 (large sample size) based on 10,000 Monte Carlo simulations. Moreover, the setting of $(\alpha_0, \alpha_1) = (0.001, 0.06), (0.004, 0.06), (0.008, 0.06)$ in large sample size K = 100 shows the change of performance of the estimates when the accelerating factor has more influence on the reliability.

Given (α_0, α_1) and K, at each (x_i, IT_i) , where $i = 1, 2, \ldots, 9$, K actual lifetimes were generated from $\text{Exp}(\lambda_i)$, where $\lambda_i = \alpha_0 e^{\alpha_1 x_i}$, of which the number of failures with lifetimes before the inspection time IT_i was recorded as n_i . Then, we determined the parameter estimate $(\hat{\alpha}_0, \hat{\alpha}_1)$, and then used it to estimate the reliabilities $(\hat{R}(10), \hat{R}(20), \hat{R}(30))$ at times $t = \{10, 20, 30\}$ and the mean lifetime $\widehat{E(T)}$, by using the EM algorithm with an initial value of (0.001, 0). The estimates from the Bayesian approach were determined next. The MCMC method was used to simulate the posterior distribution, in which the Metropolis-Hastings algorithm by using lognormal distribution for (α_0, α_1) was performed to simulate a sequence of 100,000 random variables iteratively. To approximate the posterior distribution of (α_0, α_1) , a sample of size 990 was obtained by discarding the first 1,000 iterations and then choosing 1 sample in every 100 iterations so as to reduce the correlation between the iterated samples. With the precision of the prior belief $c^2 = 0.01$ for low and moderate reliabilities and $c^2 = 0.001$ for high reliability, the prior belief \hat{p}_i were generated from the Beta (α_i^*, β_i^*) distribution. Also, an initial value of $(\alpha_0^{(0)}, \alpha_1^{(0)})$ was set for (α_0, α_1) . On the other hand, the other prior belief \tilde{p}_i was calculated from the data directly by using Eq. (2.26). The estimates obtained from the EM algorithm were taken in this case as the initial value of $(\alpha_0^{(0)}, \alpha_1^{(0)})$. In both cases under the Bayesian approach, the standard deviations of the lognormal distribution for $\alpha_0^{(r)}$ and $\alpha_1^{(r)}$ were taken to be 0.001 and 0.01, respectively.

To compare the methods based on the EM algorithm and the Bayesian approach in the case of small, moderate and large sample sizes, we computed the bias and mean square errors (MSE) of the estimates of the parameters and these are presented in Tables 2.3 to 2.11. In general, we note that, for both methods, the bias and MSE both become small when the sample size increases. Moreover, the MSE of the estimate of reliability increases with time due to the fact that fewer observations will be observed in this case. Except for the case of the estimation of the mean lifetime $\widehat{E(T)}$, the MSE of the estimates obtained from the EM algorithm is usually smaller than those obtained from the Bayesian approach incorporated with \hat{p}_i , but greater than those obtained from the Bayesian approach incorporated with \hat{p}_i . In most cases, the Bayesian estimates with \hat{p}_i yields, as expected, estimates that are close to the true parameter values with small MSE.

Tables 2.3 to 2.8 show that the estimates based on the Bayesian approach with \hat{p}_i and the EM algorithm are both quite good even in the case of small samples. The difference between the values of the MSE of these two is small for the cases of moderate and low reliabilities. When the product is of high reliability, the Bayesian approach with \hat{p}_i provides more accurate estimates than the EM algorithm. Also, the estimation based on the Bayesian method with \tilde{p}_i is not satisfactory in this case since the information used is not quite reliable in the case of small sample sizes, and so the estimates obtained by this method are far from the true parameter values. Furthermore, a comparison of the Bayesian approach by using \hat{p}_i and \tilde{p}_i shows that the precision of the estimates in this approach depends critically on a reliable prior belief. Therefore, without a proper prior belief with precision, it may not be good to adopt estimation based on the Bayesian method. In addition, the estimation based on all methods are quite similar and satisfactory when the sample size is sufficiently large. Moreover, the EM algorithm provides the most precise estimation for low, moderate and even high reliabilities. It is worth noting that, with an accurate prior belief \tilde{p}_i , the estimation based on the Bayesian approach is significantly improved in the cases of moderate and large sample sizes since in this case \tilde{p}_i converges quickly to the true parameter values, thus becoming quite reliable.

We also observe that the EM algorithm is generally not suitable for the estimation of the mean failure time. The mean lifetime obtained from the EM algorithm is always an overestimate of the true mean lifetime. In the case of small sample sizes, except the Bayesian method with a reliable \hat{p}_i , the standard deviations of estimates from the other two methods for high reliability situation is unduly large. From the simulation study, we observe that for moderate and large sample

Bias				
$\alpha_0 = 0.008$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i	
K = 10	2.673E-03	4.768E-03	8.492E-03	
K = 50	4.197E-04	1.472E-03	1.198E-03	
K = 100	2.013E-04	8.005E-04	5.756E-04	
$\alpha_1 = 0.05$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i	
K = 10	1.803E-03	-3.863E-03	-4.248E-03	
K = 50	4.003E-04	-1.362E-03	-8.488E-04	
K = 100	2.293E-04	-7.368E-04	-4.043E-04	
		MSE		
$\alpha_0 = 0.008$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i	
K = 10	1.131E-04	8.751E-05	3.293E-04	
K = 50	1.015E-05	1.172E-05	1.375E-05	
K = 100	4.688E-06	5.302E-06	5.574E-06	
$\alpha_1 = 0.05$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i	
K = 10	3.666E-04	1.576E-04	8.506E-04	
K = 50	6.425E-05	5.221E-05	6.750E-05	
K = 100	3.263E-05	2.932E-05	3.406E-05	

Table 2.3: Values of bias and mean square errors of the estimates of the model parameters in the case of low reliability for different sample sizes.

Table 2.4: Values of bias of the estimates of the reliabilities and the mean lifetime in the case of low reliability for different sample sizes.

	Bias					
R(10) = 0.756	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i			
K = 10	-7.470E-03	-2.909E-02	-4.129E-02			
K = 50	-1.135E-03	-1.083E-02	-7.503E-03			
K = 100	-4.812E-04	-6.070E-03	-3.709E-03			
R(20) = 0.572	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i			
K = 10	-4.054E-03	-3.706E-02	-4.861E-02			
K = 50	-3.939E-04	-1.419E-02	-9.293E-03			
K = 100	-6.232E-05	-7.973E-03	-4.615E-03			
R(30) = 0.433	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i			
K = 10	3.049E-03	-3.513E-02	-4.214E-02			
K = 50	1.033E-03	-1.371E-02	-8.294E-03			
K = 100	6.796E-04	-7.709E-03	-4.125E-03			
E(T) = 35.81	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i			
K = 10	3.883E+00	-1.332E+00	-6.267E-02			
K = 50	7.229E-01	-4.684E-01	5.202E-02			
K = 100	3.798E-01	-2.400E-01	3.656E-02			

MSE					
R(10) = 0.756	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	7.246E-03	3.982E-03	1.021E-02		
K = 50	1.323E-03	1.185E-03	1.463E-03		
K = 100	6.657E-04	6.412E-04	7.075E-04		
R(20) = 0.572	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	1.506E-02	7.538E-03	1.797E-02		
K = 50	2.969E-03	2.531E-03	3.175E-03		
K = 100	1.510E-03	1.410E-03	1.578E-03		
R(30) = 0.433	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	1.846E-02	8.328E-03	1.915E-02		
K = 50	3.780E-03	3.076E-03	3.922E-03		
K = 100	1.934E-03	1.757E-03	1.990E-03		
E(T) = 35.81	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	3.159E + 02	7.316E+01	2.241E+02		
K = 50	4.077E+01	2.908E + 01	3.967E + 01		
K = 100	$1.993E{+}01$	1.692E + 01	1.984E + 01		

Table 2.5: Values of mean square errors of the estimates of the reliabilities and the mean lifetime in the case of low reliability for different sample sizes.

Bias				
$\alpha_0 = 0.004$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i	
K = 10	1.604E-03	2.883E-03	5.545E-03	
K = 50	3.355E-04	1.322E-03	8.403E-04	
K = 100	1.558E-04	5.341E-04	3.885E-04	
$\alpha_1 = 0.05$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i	
K = 10	1.801E-03	-5.019E-03	-6.276E-03	
K = 50	9.101E-05	-3.532E-03	-1.452E-03	
K = 100	6.277E-05	-1.187E-03	-6.891E-04	
		MSE		
$\alpha_0 = 0.004$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i	
K = 10	3.794 E-05	2.972E-05	1.313E-04	
K = 50	3.505E-06	6.399E-06	4.897E-06	
K = 100	1.482E-06	1.761E-06	1.773E-06	
$\alpha_1 = 0.05$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i	
K = 10	4.064E-04	1.612E-04	4.168E-04	
K = 50	7.642E-05	8.308E-05	7.795E-05	
K = 100	3.691E-05	3.314 E-05	3.749E-05	

Table 2.6: Values of bias and mean square errors of the estimates of the model parameters in the case of moderate reliability for different sample sizes.

Table 2.7: Values of bias of the estimates of the reliabilities and the mean lifetime in the case of moderate reliability for different sample sizes.

	Bias					
R(10) = 0.870	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i			
K = 10	-5.846E-03	-2.166E-02	-3.319E-02			
K = 50	-1.905E-03	-9.001E-03	-7.023E-03			
K = 100	-8.761E-04	-4.992E-03	-3.366E-03			
R(20) = 0.756	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i			
K = 10	-6.931E-03	-3.444E-02	-5.066E-02			
K = 50	-2.703E-03	-1.433E-02	-1.123E-02			
K = 100	-1.233E-03	-8.145E-03	-5.404E-03			
R(30) = 0.658	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i			
K = 10	-5.031E-03	-4.106E-02	-5.798E-02			
K = 50	-2.742E-03	-1.703E-02	-1.340E-02			
K = 100	-1.232E-03	-9.934E-03	-6.470E-03			
E(T) = 71.63	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i			
K = 10	1.171E+01	-2.969E+00	5.613E-01			
K = 50	1.569E + 00	-1.894E-01	-2.205E-01			
K = 100	7.859E-01	-7.248E-01	-6.729E-02			

MSE					
R(10) = 0.870	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	3.237E-03	1.794E-03	5.054E-03		
K = 50	6.093E-04	6.857E-04	6.867E-04		
K = 100	2.907E-04	2.826E-04	3.097E-04		
R(20) = 0.756	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	9.102E-03	4.792E-03	1.275E-02		
K = 50	1.810E-03	1.961E-03	1.992E-03		
K = 100	8.719E-04	8.302E-04	9.183E-04		
R(30) = 0.658	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	1.464E-02	7.289E-03	1.853E-02		
K = 50	3.034E-03	3.169E-03	3.264E-03		
K = 100	1.473E-03	1.375E-03	1.534E-03		
E(T) = 71.63	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	2.106E + 03	3.360E + 02	1.262E + 03		
K = 50	2.246E+02	1.930E + 02	2.076E + 02		
K = 100	1.046E+02	8.440E+01	1.016E + 02		

Table 2.8: Values of mean square errors of the estimates of the reliabilities and the mean lifetime in the case of moderate reliability for different sample sizes.

Bias					
$\alpha_0 = 0.001$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	1.663E-03	5.626E-04	8.680E-03		
K = 50	2.212E-04	3.209E-04	6.794E-04		
K = 100	1.168E-04	2.364E-04	3.234E-04		
$\alpha_1 = 0.05$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	3.418E-03	-4.257E-03	-2.086E-02		
K = 50	4.863E-04	-2.911E-03	-4.396E-03		
K = 100	1.836E-04	-2.249E-03	-2.321E-03		
		MSE			
$\alpha_0 = 0.001$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	4.371E-05	6.355E-06	3.813E-04		
K = 50	8.634E-07	3.543E-07	1.831E-06		
K = 100	3.408E-07	2.241E-07	5.375E-07		
$\alpha_1 = 0.05$	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	1.295E-03	9.420E-05	1.491E-03		
K = 50	2.058E-04	6.335E-05	2.145E-04		
K = 100	1.073E-04	4.918E-05	1.105E-04		

Table 2.9: Values of bias and mean square errors of the estimates of the model parameters in the case of high reliability for different sample sizes.

Table 2.10: Values of bias of the estimates of the reliabilities and the mean lifetime in the case of high reliability for different sample sizes.

Bias					
R(10) = 0.966	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	-6.527E-03	-5.224E-03	-3.402E-02		
K = 50	-1.239E-03	-3.481E-03	-6.163E-03		
K = 100	-7.452E-04	-2.631E-03	-3.246E-03		
R(20) = 0.933	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	-1.161E-02	-9.881E-03	-6.193E-02		
K = 50	-2.248E-03	-6.611E-03	-1.162E-02		
K = 100	-1.366E-03	-5.003E-03	-6.142E-03		
R(30) = 0.901	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	-1.545E-02	-1.402E-02	-8.476E-02		
K = 50	-3.051E-03	-9.416E-03	-1.642E-02		
K = 100	-1.874E-03	-7.134E-03	-8.717E-03		
E(T) = 286.50	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	3.326E+02	-1.209E+01	8.447E+01		
K = 50	2.486E+01	-7.977E+00	-1.855E+00		
K = 100	1.161E+01	-6.161E+00	-1.642E+00		

MSE					
R(10) = 0.966	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	9.926E-04	1.014E-04	2.793E-03		
K = 50	1.438E-04	5.882E-05	1.990E-04		
K = 100	7.280E-05	4.223E-05	8.840E-05		
R(20) = 0.933	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	3.457E-03	3.655E-04	9.083E-03		
K = 50	5.297E-04	2.156E-04	7.205E-04		
K = 100	2.698E-04	1.555E-04	3.246E-04		
R(30) = 0.901	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	6.809E-03	7.422E-04	1.677E-02		
K = 50	1.098E-03	4.446E-04	1.469E-03		
K = 100	5.627E-04	3.220E-04	6.708E-04		
E(T) = 286.50	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
K = 10	9.106E + 06	3.760E + 03	6.143E+06		
K = 50	1.457E + 04	2.859E + 03	1.106E + 04		
K = 100	6.244E+03	2.259E + 03	5.458E + 03		

Table 2.11: Values of mean square errors of the estimates of the reliabilities and the mean lifetime in the case of high reliability for different sample sizes.

Table 2.12: Values of bias and mean square errors of the estimates of the model parameters, the reliabilities, and the mean lifetime in the case of high reliability with $\alpha_1 = 0.06$ for large sample sizes.

Bias					
	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
$\alpha_0 = 0.001$	1.000E-04	1.000E-04	2.000E-04		
$\alpha_1 = 0.06$	2.000E-04	-1.400E-03	-1.600E-03		
R(10) = 0.956	-5.726E-04	-2.073E-03	-2.873E-03		
R(20) = 0.914	-9.659E-04	-3.866E-03	-5.266E-03		
R(30) = 0.874	-1.296E-03	-5.396E-03	-7.396E-03		
E(T) = 223.13	6.510E+00	-2.870E+00	-8.700E-01		
		MSE			
	EM algorithm	Bayesian with \hat{p}_i	Bayesian with \tilde{p}_i		
$\alpha_0 = 0.001$	2.600E-07	1.000E-07	2.900E-07		
$\alpha_1 = 0.06$	7.229E-05	2.900E-05	7.481E-05		
R(10) = 0.956	7.954E-05	3.794E-05	9.289E-05		
R(20) = 0.914	2.933E-04	1.382E-04	3.340E-04		
R(30) = 0.874	5.970E-04	2.819E-04	6.797E-04		
E(T) = 223.13	2.417E+03	8.504E + 02	2.199E+03		

sizes, the mean lifetime is estimated well by all the methods in low, moderate and even high reliability situations, but the estimation based on the EM algorithm is less efficient than that based on the Bayesian method with \hat{p}_i and \tilde{p}_i in most situations considered. On the other hand, when α_1 is changed from 0.05 to 0.06, Table 2.12 shows that the performance of all methods are quite similar to that in Tables 2.9, 2.10 and 2.11.

To conclude, the EM algorithm is preferable for the estimation of the model parameters and the reliability, while the Bayesian method with \tilde{p}_i is better for the estimation of the mean lifetime in moderate and large sample sizes. In small sample size, the Bayesian method with a reliable prior belief provides the best estimation. If a reliable prior belief is unavailable, the EM algorithm is an useful alternative approach for the estimation of the parameters as well as the reliability. But, for the estimation of the mean lifetime, the Bayesian approach with \tilde{p}_i seems to be a better choice.

2.6 Concluding Remarks

In this Chapter, we have developed the EM algorithm for one-shot device testing with an accelerating factor of temperature. Fan *et al.* [19] proposed the Bayesian approach incorporating a reliable prior belief \hat{p}_i for such a testing procedure in case of small sample sizes. Unfortunately, the reliable prior belief may not always be available in practice and thus it becomes somewhat subjective. Even though a reasonable and common prior belief, $\tilde{p}_i = 1 - n_i/K$, is available in the data, it is not reliable in the case of small sample sizes. We have compared the EM algorithm method with the Bayesian approach using the two priors through Monte Carlo simulations. The simulation results suggest that the EM algorithm is generally better for the estimation of the model parameters and the reliability since the Bayesian method without a reliable prior belief may not provide accurate estimation. For the estimation of the mean lifetime, however, the EM algorithm usually results in overestimation while the Bayesian method incorporating a reliable belief performs well in this case.

The estimates by the EM algorithm are not attainable in the situation when $n_i = 0$ for all i = 1, 2, 3, ..., I; that is, in the case when all devices tested in higher temperatures are successfully detonated. In such a situation, we obtain no information on the reliability. For this reason, the EM algorithm does not work well for devices with very high reliability. For avoiding the situation of no failure occurring in the case of very high reliability, one may either use a much larger sample size or choose to shift the inspection time further to the right.

Chapter 3

Exponential Lifetime Distribution with Multiple-Stress Model

3.1 Introduction

In the preceding Chapter, a single-stress relationship under exponential distribution has been discussed. As opposed to a single-stress test by using a high stress level so as to attain the aging within a limited time, some ALTs involve two or more stress factors. For example, Morris [40] analyzed a battery data from destructive life-tests in which both mildly high temperature and light load were applied as stress. Multiple-stress ALTs would enable us to attain adequate failure data in a relatively short period of time without requiring any of the stress factors to be set at very high levels. If maintaining a stress factor at high stress level for testing purposes is expensive, one could introduce several stress factors set at slightly elevated stress levels causing more devices to fail than would under a single-stress test. For this reason, multiple-stress models are better suited for the prediction of lifetimes of electronic products, subjected to electrical, thermal and mechanical stresses; see, for example, [4], [6], [48], [57], [60], [61], and [67].

3.2 Model Description

Here, we assume that the devices subjected to J types of stress factors under the *i*-th testing condition have a failure rate λ_i that is expressed through a loglinear link function

$$\lambda_i = \exp\left(\sum_{j=0}^J a_j x_{ij}\right),\tag{3.1}$$

where $x_{i0} \equiv 1$ and x_{ij} is the level of the *j*-th stress under the *i*-th testing condition. Let us denote $\boldsymbol{a} = \{a_j, j = 0, 1, ..., J\}$ and $\boldsymbol{x}_i = \{x_{ij}, j = 0, 1, ..., J\}$. The corresponding pdf and cdf of the lifetime of the devices, under the exponential lifetime distribution, are obtained to be

$$f(t; \boldsymbol{x}_i) = \exp\left(\sum_{j=0}^J a_j x_{ij}\right) \exp\left(-t \exp\left(\sum_{j=0}^J a_j x_{ij}\right)\right), \qquad t \ge 0, \qquad (3.2)$$

and

$$F(t; \boldsymbol{x}_i) = 1 - \exp\left(-t \exp\left(\sum_{j=0}^J a_j x_{ij}\right)\right), \qquad t \ge 0, \tag{3.3}$$

respectively.

Then, the reliability at time t and the mean lifetime under normal operating

conditions $\boldsymbol{x}_0 = \{x_{0j}, j = 0, 1, \dots, J\}$ are given by

$$R(t; \boldsymbol{x}_0) = 1 - F(t; \boldsymbol{x}_0) = \exp\left(-t \exp\left(\sum_{j=0}^J a_j x_{0j}\right)\right), \qquad t \ge 0, \qquad (3.4)$$

and

$$E(T) = \frac{1}{\lambda_0} = \exp\left(-\sum_{j=0}^J a_j x_{0j}\right),$$
 (3.5)

respectively.

3.3 Point Estimation of Parameters of Interest

3.3.1 EM Algorithm Based on One-Step Newton-Raphson Method

Consider the log-likelihood function of the complete data (along with the conditional expectation) given by

$$\ell_{c} = \sum_{i=1}^{I} \sum_{k=1}^{K_{i}} \left(\sum_{j=0}^{J} a_{j} x_{ij} - t_{ik} \exp\left(\sum_{j=0}^{J} a_{j} x_{ij}\right) \right)$$
$$= \sum_{i=1}^{I} \left(K_{i} \sum_{j=0}^{J} a_{j} x_{ij} \right) - \sum_{i=1}^{I} \left(\left(\sum_{k=1}^{K_{i}} t_{ik}\right) \exp\left(\sum_{j=0}^{J} a_{j} x_{ij}\right) \right).$$
(3.6)

Let us denote $(\mathbf{K}, \mathbf{IT}, \mathbf{a}', \mathbf{x}) = \{(K_i, IT_i, a'_j, \mathbf{x}_i), i = 1, 2, ..., I\}$, where a'_j are the current estimates in the E-step. The expected log-likelihood function of the complete data is then obtained as

$$Q(\boldsymbol{a}|\boldsymbol{a}') = E(\ell_c|\boldsymbol{K}, \boldsymbol{IT}, \boldsymbol{a}', \boldsymbol{x})$$
$$= \sum_{i=1}^{I} \left(K_i \sum_{j=0}^{J} a_j x_{ij} \right) - \sum_{i=1}^{I} \left(K_i T_i^* \exp\left(\sum_{j=0}^{J} a_j x_{ij}\right) \right).$$
(3.7)

In the M-step, to update estimates of the parameters, for j = 0, 1, ..., J, the following first-order derivative with respect to a_j to maximize the quantity $Q(\boldsymbol{a}|\boldsymbol{a}')$ in Eq. (3.7) is then obtained as

$$\frac{\partial Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_j} = \sum_{i=1}^{I} K_i x_{ij} - \sum_{i=1}^{I} \left(K_i x_{ij} T_i^* \exp\left(\sum_{j=0}^{J} a_j x_{ij}\right) \right).$$
(3.8)

Since the lifetimes of the devices t_{ik} are censored, the conditional expectation in the E-step is straightforward and is obtained as follows:

$$T_{i}^{*} = E(T_{ik} | \boldsymbol{K}, \boldsymbol{IT}, \boldsymbol{a}', \boldsymbol{x})$$

$$= \frac{n_{i} \int_{0}^{IT_{i}} t\lambda_{i}' \exp(-\lambda_{i}'t) dt}{K_{i} \int_{0}^{IT_{i}} \lambda_{i}' \exp(-\lambda_{i}'t) dt} + \frac{(K_{i} - n_{i}) \int_{IT_{i}}^{\infty} t\lambda_{i}' \exp(-\lambda_{i}'t) dt}{K_{i} \int_{IT_{i}}^{\infty} \lambda_{i}' \exp(-\lambda_{i}'t) dt}$$

$$= \left(\frac{n_{i}}{K_{i}}\right) \frac{1 - \exp(-\lambda_{i}'IT_{i}) - \lambda_{i}'IT_{i} \exp(-\lambda_{i}'IT_{i})}{\lambda_{i}'(1 - \exp(-\lambda_{i}'IT_{i}))}$$

$$+ \left(1 - \frac{n_{i}}{K_{i}}\right) \frac{\exp(-\lambda_{i}'IT_{i}) + \lambda_{i}'IT_{i} \exp(-\lambda_{i}'IT_{i})}{\lambda_{i}' \exp(-\lambda_{i}'IT_{i})}$$

$$= \frac{1}{\lambda_{i}'} + IT_{i} - \left(\frac{n_{i}IT_{i}}{K_{i}(1 - \exp(-\lambda_{i}'IT_{i})}\right), \qquad (3.9)$$

where $\lambda'_i = \exp\left(\sum_{j=0}^J a'_j x_{ij}\right)$.

For finding the MLEs in every M-step, we need to set the first derivatives in Eq. (3.8) to 0 and solve the system of equations. Since there is no closed-form solution for these equations, the Newton-Raphson method is usually employed for finding the solution to this system of equations at each M-step, but this could result in very intensive computation. For this reason, instead of implementing the usual Newton-Raphson method ([23],[42],[47]), the one-step Newton-Raphson method could be used and is as follows. The EM algorithm based on one-step Newton-

Raphson method, described by McLachlan and Krishnan [36], is often adopted for the determination of the MLEs due to its convenience as it helps in reducing the amount of computation and also assists in accelerating the convergence of the EM algorithm.

In a longitudinal study involving Cox model, while using the EM algorithm, Wulfsohn and Tsiatis [65] adopted the one-step Newton-Raphson method for estimating a parameter that has no closed-form in the M-step. In the present situation, let us denote

$$\mathbf{I} = \begin{bmatrix} \frac{\partial Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_0} & \frac{\partial Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_1} & \dots & \frac{\partial Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_J} \end{bmatrix}_{\boldsymbol{a}=\boldsymbol{a}'}^T, \quad (3.10)$$
$$\mathbf{X} = \begin{bmatrix} a'_0 & a'_1 & \dots & a'_J \end{bmatrix}, \quad (3.11)$$

and

$$\mathbf{J} = - \begin{bmatrix} \frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_0^2} & \frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_0 \partial a_1} & \cdots & \frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_0 \partial a_J} \\ \frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_1 \partial a_0} & \frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_1^2} & \cdots & \frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_1 \partial a_J} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_J \partial a_0} & \frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_J \partial a_1} & \cdots & \frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_J^2} \end{bmatrix}_{\boldsymbol{a}=\boldsymbol{a}'}, \quad (3.12)$$

where

$$\frac{\partial^2 Q(\boldsymbol{a}|\boldsymbol{a}')}{\partial a_p \partial a_q} = -\sum_{i=1}^{I} \left(K_i x_{ip} x_{iq} T_i^* \exp\left(\sum_{j=0}^{J} a_j x_{ij}\right) \right), \qquad p, q = 0, 1, \dots, J.$$
(3.13)

Then, we get the updated parameters as

$$\hat{\mathbf{X}} = \mathbf{J}^{-1}\mathbf{I} + \mathbf{X}^T.$$
(3.14)

3.3.2 The Choice of Initial Guess for the EM Algorithm

The most challenging part of this approach is the choice of initial values for the parameters since the data is of binary form. The iterations are less likely to converge if the initial values are not chosen carefully. Method of moments is a common method to provide the initial values for the parameters, but this would require information on the actual lifetimes of devices which we do not have in the present setting of one-shot device testing data. Instead, due to the log-linear form of the covariates, the initial values can be obtained by the least-squares method as

$$\hat{\mathbf{X}} = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{Y}^{(0)}, \qquad (3.15)$$

where

$$\mathbf{Y}^{(\mathbf{0})} = \begin{bmatrix} \log\left(\lambda_1^{(0)}\right) & \log\left(\lambda_2^{(0)}\right) & \cdots & \log\left(\lambda_I^{(0)}\right) \end{bmatrix}^T, \quad (3.16)$$

$$\lambda_i^{(0)} = -\frac{\log\left(1 - \frac{n_i}{K_i}\right)}{IT_i}, \qquad i = 1, 2, ..., I,$$
(3.17)

and

$$\mathbf{A} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1J} \\ 1 & x_{21} & \cdots & x_{2J} \\ \vdots & \vdots & & \vdots \\ 1 & x_{I1} & \cdots & x_{IJ} \end{bmatrix}.$$
 (3.18)

In this case, when $n_i = 0$ or K_i , an adjustment is needed in the above expression and we do this by replacing $n_i = 0$ with $n_i = \frac{1}{10K_i}$ and $n_i = K_i$ with $n_i = K_i - \frac{1}{10K_i}$ for the initial values of the parameters. Again, given the normal stress levels $\boldsymbol{x}_0 = \{x_{0j}, j = 0, 1, ..., J\}$ and the MLEs of the model parameter $\hat{\boldsymbol{a}} = \{\hat{a}_j, j = 0, 1, ..., J\}$, the MLEs of the reliability at a mission time t and the mean lifetime, under the assumption of exponentiality for the lifetimes, are given by

$$\hat{R}(t; \boldsymbol{x}_0) = \exp\left(-t \exp\left(\sum_{j=0}^J \hat{a}_j x_{0j}\right)\right)$$
(3.19)

and

$$\widehat{E(T)} = \exp\left(-\sum_{j=0}^{J} \hat{a}_j x_{0j}\right), \qquad (3.20)$$

respectively.

3.3.3 Jackknife Method for Bias Reduction

Since the MLEs of the parameters tend to be biased in this case, we propose bias-corrected estimation by using the jackknife technique. Jackknife method is a systematic re-sampling technique, omitting one observation at a time from the original sample, commonly employed for reducing the bias in the estimation. This method is useful not only for the estimation of model parameters but also for the estimation of the reliability at a specific mission time and the mean lifetime of the devices. In the case of one-shot device testing data, there is an advantage since the calculation would not take much time for the jackknife technique. Following the jackknife method presented in [12], suppose the MLE of interest $\hat{\theta}$ is obtained from the original data. For i = 1, 2, ..., I, a device failed in the testing experiment at inspection time IT_i is deleted from the data, and so both n_i and K_i are reduced by 1. Then, the MLE of θ based on the reduced data is obtained, and denoted by $\hat{\theta}_F^{(-i)}$. Similarly, for the cases when a device tested successfully at inspection time IT_i is deleted from the data, the MLE of θ based on the reduced data (by reducing only K_i by 1) is also obtained, and denoted by $\hat{\theta}_S^{(-i)}$.

Let $N = \sum_{i=1}^{I} K_i$. Then, the jackknifed bias of the estimator is then given by

$$bias(\hat{\theta}) = -\frac{N-1}{N} \sum_{i=1}^{I} \left(n_i(\hat{\theta} - \hat{\theta}_F^{(-i)}) + (K_i - n_i)(\hat{\theta} - \hat{\theta}_S^{(-i)}) \right).$$
(3.21)

From the above expression, we note that it is not necessary to compute $\hat{\theta}_F^{(-i)}$ when $n_i = 0$. Hence, the jackknifed estimator of θ is simply

$$\hat{\theta}_{JK} = \hat{\theta} - bias(\hat{\theta}) = N\hat{\theta} - (N-1)\bar{\hat{\theta}}, \qquad (3.22)$$

where

$$\bar{\hat{\theta}} = \frac{\sum_{i=1}^{I} \left(n_i \hat{\theta}_F^{(-i)} + (K_i - n_i) \hat{\theta}_S^{(-i)} \right)}{N}.$$
(3.23)

It is important to mention that the jackknifed estimates of the reliability and the mean lifetime could be outside the range of their supports when the sample size is small. In this case, therefore, the estimates need to be corrected suitably at the lower and upper ends.

3.4 Interval Estimation of Parameters of Interest

In addition to the point estimation, we also examine three different confidence intervals for the parameters of interest constructed by using the asymptotic properties of MLEs, the jackknife method, and the parametric bootstrap method. Furthermore, a transformation technique is also suggested which is particularly useful when the distribution of the underlying pivoting quantity is skewed. The efficiency of these confidence intervals are then assessed by means of Monte Carlo simulations.

3.4.1 Use of Observed Fisher Information Matrix

When the EM algorithm is employed for finding the MLEs based on censored data, in order to extract the observed information matrix for the calculation of the asymptotic variance-covariance matrix of the MLEs, the Missing Information Principle developed by Louis [34] is commonly used. The observed information matrix requires the complete information matrix as well as the missing information matrix. These matrices are given by

$$I_{complete} = -E\left[\frac{\partial^2(\ell_c(\boldsymbol{\theta}))}{\partial \boldsymbol{\theta}^2}\right]$$
(3.24)

and

$$I_{missing} = -\sum_{i=1}^{I} \sum_{k=1}^{K_i} E\left[\frac{\partial^2(\log(f(t_{ik}|\mathbf{z},\boldsymbol{\theta})))}{\partial \boldsymbol{\theta}^2}\right],$$
(3.25)

respectively. Using these, we will then obtain the observed information matrix as

$$I_{obs} = I_{complete} - I_{missing}.$$
 (3.26)

Ng *et al.* [47] and Nandi and Dewan [42] used this method for deriving the Fisher information matrix under different censoring schemes. On the other hand, following
the work of Dempster *et al.* [13], a direct method for deriving the Fisher information matrix was developed by Oakes [49]. Friedl and Kauermann [22] adopted this direct procedure for computing the asymptotic variance-covariance matrix of the MLEs in generalized linear models with random effects. Since, for the case when all lifetimes are censored, the observed Fisher information matrix based on the observed likelihood function is identical to the observed information matrix obtained by the Missing Information Principle, we determine here the observed Fisher information matrix by using the log-likelihood function of observed data.

Result 3.1. Suppose the lifetime distribution has a probability density function $f(t; \theta)$ and a cumulative distribution function $F(t; \theta)$. For the case when all lifetimes are censored, the observed Fisher information matrix based on the observed likelihood function is identical to the observed information matrix obtained by the Missing Information Principle.

Proof: Given a data with a sequence of inspection times $0 < IT_1 < IT_2 < \cdots < IT_{I-1}$ and the corresponding numbers of failures in each time slot, $\{n_1, n_2, \ldots, n_I\}$, and K observed failure times, $t_k, k = 1, 2, \ldots, K$, the log-likelihood function is then given by

$$\ell(\boldsymbol{\theta}) = n_1 \log F(IT_1; \boldsymbol{\theta}) + \sum_{i=2}^{I-1} n_i \log(F(IT_i; \boldsymbol{\theta}) - F(IT_{i-1}; \boldsymbol{\theta})) + n_I \log(1 - F(IT_{I-1}; \boldsymbol{\theta})) + \sum_{k=1}^{K} \log(f(t_k; \boldsymbol{\theta})) + \text{constant}, \qquad (3.27)$$

where n_1, n_2, \ldots, n_I and t_1, t_2, \ldots, t_K are random variables.

On the other hand, the log-likelihood function for complete data is given by

$$\ell_{complete} = \sum_{i=1}^{I} \sum_{j=1}^{n_i} \log(f(t_j; \boldsymbol{\theta})) + \sum_{k=1}^{K} \log(f(t_k; \boldsymbol{\theta})) + \text{constant}, \quad (3.28)$$

while the log-likelihood function of conditional distribution for missing data is given by

$$\ell_{missing} = \sum_{j=1}^{n_1} (\log(f(t_j; \boldsymbol{\theta})) - \log(F(IT_1; \boldsymbol{\theta}))) + \sum_{i=2}^{I-1} \sum_{j=1}^{n_i} (\log(f(t_j; \boldsymbol{\theta})) - \log(F(IT_i; \boldsymbol{\theta}) - F(IT_{i-1}; \boldsymbol{\theta}))) + \sum_{i=1}^{n_I} (\log(f(t_j; \boldsymbol{\theta})) - \log(1 - F(IT_{I-1}; \boldsymbol{\theta}))) + \text{constant}) + \sum_{j=1}^{I} \log(f(t_j; \boldsymbol{\theta})) - \sum_{j=1}^{n_1} \log(F(IT_1; \boldsymbol{\theta})) + \text{constant}) = \sum_{i=2}^{I} \sum_{j=1}^{n_i} \log(f(t_j; \boldsymbol{\theta})) - \sum_{j=1}^{n_1} \log(F(IT_1; \boldsymbol{\theta})) - \sum_{i=2}^{I} \log(F(IT_i; \boldsymbol{\theta}) - F(IT_{i-1}; \boldsymbol{\theta}))) - \sum_{j=1}^{n_I} \log(F(IT_i; \boldsymbol{\theta}) - F(IT_{i-1}; \boldsymbol{\theta}))) - \sum_{j=1}^{n_I} \log(1 - F(IT_{I-1}; \boldsymbol{\theta})) + \text{constant}.$$
(3.29)

It should be noted that t_j in Eqs. (3.28) and (3.29) are also random variables, but the terms of $\log(f(t_j; \boldsymbol{\theta}))$ have canceled out by the Missing Information Principle. Hence, the observed log-likelihood function is obtained from the Missing Information Principle as follows:

$$\ell_{obs} = n_1 \log F(IT_1; \boldsymbol{\theta}) + \sum_{i=2}^{I-1} n_i \log(F(IT_i; \boldsymbol{\theta}) - F(IT_{i-1}; \boldsymbol{\theta})) + n_I \log(1 - F(IT_{I-1}; \boldsymbol{\theta})) + \sum_{k=1}^{K} \log(f(t_k; \boldsymbol{\theta})) + \text{constant}, \quad (3.30)$$

where t_1, t_2, \ldots, t_K are random variables. Therefore, for the case when all lifetimes are censored, the terms of $\sum_{k=1}^{K} \log(f(t_k; \boldsymbol{\theta}))$ in Eqs. (3.27) and (3.30) disappear.

Moreover, it is easy to show that the observed information matrix from the Missing Information Principle is different from the Fisher information matrix based on the observed likelihood function, but is identical to the observed Fisher information matrix based on the observed likelihood function.

From the above, we also note that the observed log-likelihood function from the Missing Information Principle is a mixture of Fisher information and observed Fisher information. The observed numbers of failures are used for n_1, n_2, \ldots, n_I in the part of censored data, and in contrast we take expectations $E\left[\frac{\partial^2 log(f(t_k; \boldsymbol{\theta}))}{\partial \boldsymbol{\theta}^2}\right]$ in the part of observed data.

For the case of one-shot device testing data, due to the fact that all failure times are censored, the observed information matrix by Missing Information Principle is identical to the observed Fisher information matrix obtained from the log-likelihood function conditional on \mathbf{z} . The observed log-likelihood function is given by

$$\ell = \sum_{i=1}^{I} \left[n_i \log \left(1 - e^{-IT_i \lambda_i} \right) - \left(K_i - n_i \right) \left(IT_i \lambda_i \right) \right].$$
(3.31)

We then have the observed Fisher information matrix corresponding to the model parameters as

$$\mathbf{I}_{obs} = \left(-\frac{\partial^2 \ell}{\partial a_p \partial a_q}\right)_{(p,q)},\tag{3.32}$$

where

$$\frac{\partial^2 \ell}{\partial a_p \partial a_q} = \sum_{i=1}^{I} \frac{n_i x_{ip} x_{iq} \lambda_i I T_i e^{-\lambda_i I T_i} (1 - \lambda_i I T_i - e^{-\lambda_i I T_i})}{(1 - e^{-\lambda_i I T_i})^2} - \sum_{i=1}^{I} (K_i - n_i) x_{ip} x_{iq} \lambda_i I T_i.$$
(3.33)

The asymptotic variance-covariance matrix of the MLEs of the model parameters, \mathbf{V} , can be obtained by inverting the observed Fisher information matrix \mathbf{I}_{obs} of the MLEs of the model parameters.

Furthermore, we can use the delta method along with this information matrix to find the variance of the estimates of the reliability at mission time t and the mean lifetime at stress level \boldsymbol{x}_0 . From Eqs. (3.19) and (3.20), we have

$$\frac{\partial R(t; \boldsymbol{x}_0)}{\partial a_j} = -t\lambda_0 x_j \exp\left(-t\lambda_0\right)$$
(3.34)

and

$$\frac{\partial E(T)}{\partial a_j} = -x_j E(T). \tag{3.35}$$

Now, defining

$$\mathbf{P}_{\mathbf{R}} = \left[\frac{\partial \hat{R}(t; \boldsymbol{x}_0)}{\partial a_0}, ..., \frac{\partial \hat{R}(t; \boldsymbol{x}_0)}{\partial a_J}\right]_{\boldsymbol{a} = \boldsymbol{a}'}^T$$
(3.36)

and

$$\mathbf{P}_{\mathbf{E}(\mathbf{T})} = \left[\frac{\partial \widehat{E(T)}}{\partial a_0}, ..., \frac{\partial \widehat{E(T)}}{\partial a_J}\right]_{\boldsymbol{a}=\boldsymbol{a}'}^T, \qquad (3.37)$$

we have the corresponding variances of the estimates of the reliability at mission time t and the mean lifetime at stress level x_0 to be

$$\mathbf{V}_{\mathbf{R}} = \mathbf{P}_{\mathbf{R}}^{T} \mathbf{V} \mathbf{P}_{\mathbf{R}}$$
(3.38)

and

$$\mathbf{V}_{\mathbf{E}(\mathbf{T})} = \mathbf{P}_{\mathbf{E}(\mathbf{T})}^{T} \mathbf{V} \mathbf{P}_{\mathbf{E}(\mathbf{T})}.$$
(3.39)

Consequently, an approximate $100(1 - \alpha)\%$ confidence interval for the parameter θ is given by

$$\left(\hat{\theta} - z_{1-\frac{\alpha}{2}}se(\hat{\theta}), \hat{\theta} + z_{1-\frac{\alpha}{2}}se(\hat{\theta})\right), \qquad (3.40)$$

where $z_{1-\frac{\alpha}{2}}$ is the $1-\frac{\alpha}{2}$ normal quantile and $se(\hat{\theta})$ is the standard error of $\hat{\theta}$. In the confidence intervals for reliability, the lower and upper bounds are not guaranteed to be bounded between 0 and 1. Therefore, a correction is necessary wherein we set the lower bound to be at least 0 and the upper bound to be at most 1. Similarly, a correction is also necessary on the confidence interval for the mean lifetime wherein we set the lower bound to be at least 0.

3.4.2 Use of Jackknife Technique

Due to the non-linear form of the quantities of the reliability and the mean lifetime, re-sampling technique is used for the computation of the standard errors of the estimates, which are then subsequently used in the construction of confidence intervals. It is important to mention here that the jackknife technique has been used for variance estimation in some other contexts as well; for example, Gini index [12] and harmonic mean of rate-based metric [51] have been investigated in this manner. Now, we consider the variance estimation of the MLE of a parameter of interest by using the jackknife technique. As we described before, the jackknifed point estimator, based on the reduced data, produces a set of MLEs of the parameter. From this set of estimates, the jackknifed variance can be calculated as

$$\hat{v}_{JK} = \frac{N-1}{N} \sum_{i=1}^{I} \left\{ n_i \left(\hat{\theta}_F^{(-i)} - \bar{\hat{\theta}} \right)^2 + (K_i - n_i) \left(\hat{\theta}_S^{(-i)} - \bar{\hat{\theta}} \right)^2 \right\}.$$
 (3.41)

Hence, given the bias-corrected estimate $\hat{\theta}_{JK}$, the $100(1-\alpha)\%$ confidence interval for θ_{JK} is then given by

$$\left(\hat{\theta}_{JK} - z_{1-\frac{\alpha}{2}}\sqrt{\hat{v}_{JK}}, \hat{\theta}_{JK} + z_{1-\frac{\alpha}{2}}\sqrt{\hat{v}_{JK}}\right).$$
(3.42)

As before, we may have to do corrections on the jackknifed confidence intervals for the reliability at mission time and the mean lifetime, if necessary.

3.4.3 Use of Parametric Bootstrap Method

Instead of using the jackknife technique on reduced data by deleting one observation at a time, a set of parametric bootstrap data can be generated according to the supposed parametric model based on the original data, and then these data could be used to determine the estimate of the parameter. By repeatedly performing bootstrap simulations, the corresponding distribution of the estimate can be approximated. The confidence interval can then be constructed based on the empirical distribution of the estimate of the parameter. Since the lifetime distribution of the devices is assumed to be exponential, the confidence interval for the parameter of interest can be constructed by the parametric bootstrap method as follows:

1. Find \hat{a}_j from the original data, where j = 0, 1, ..., J;

- 2. Obtain bootstrap samples $\{n_i^*, i = 1, 2, ..., I\}$ based on $\{\hat{a}_j, j = 0, 1, ..., J\}$ and the other observed data $\{K, IT, x\}$;
- 3. Find the bootstrap estimate of θ based on the bootstrap sample $\{n_i^*, i = 1, 2, ..., I\}$, denoted by $\hat{\theta}^b$;
- 4. Repeat Steps 2 and 3 B times to obtain the bootstrap estimates $\hat{\theta}^b, b = 1, 2, ..., B$;
- 5. Sort the bootstrap estimates $\hat{\theta}^b$ in ascending order, denoted by $\hat{\theta}^{[b]}, b = 1, 2, ..., B$.

Then, the $100(1-\alpha)\%$ parametric bootstrap confidence interval for the parameter θ is given by

$$\left(\hat{\theta}^{\left[\frac{\alpha}{2}(B+1)\right]}, \hat{\theta}^{\left[\left(1-\frac{\alpha}{2}\right)(B+1)\right]}\right).$$
(3.43)

It should be mentioned that the bootstrap method does not require any correction.

3.4.4 Use of Transformation Method

To avoid performing corrections involved in the asymptotic and jackknife confidence intervals when the sample size is small, a transformation technique can be suggested which is particularly useful when the distribution for the corresponding pivotal quantity is skewed. In this connection, Meeker and Escobar [37] used the logit (log-odds) transformation for approximating the confidence intervals for the reliability, and the normal approximation for the log of the mean lifetime, which ensures that the bounds for the reliability always fall between 0 and 1, and also avoids having lower bound for the mean lifetime to be negative.

For the reliability approximation, the log-odds of the estimated reliability

$$g_1 = \log\left(\frac{\hat{R}(t)}{1 - \hat{R}(t)}\right) \tag{3.44}$$

is assumed to be normally distributed. By the delta method, we then have

$$\widehat{se}(g_1) = \frac{\widehat{se}(\hat{R}(t))}{\hat{R}(t)(1 - \hat{R}(t))},$$
(3.45)

where $\hat{se}(\hat{R}(t))$ is the estimated standard deviation of $\hat{R}(t)$, and that \hat{g}_1 is then normally distributed with mean g_1 and standard deviation $se(g_1)$ which is estimated by Eq. (3.45). Consequently, the $100(1 - \alpha)\%$ confidence interval for the log-odds of the reliability g_1 is

$$\left(\hat{g}_1 - z_{1-\frac{\alpha}{2}}\widehat{s}\hat{e}(g_1), \hat{g}_1 + z_{1-\frac{\alpha}{2}}\widehat{s}\hat{e}(g_1)\right).$$
 (3.46)

By inverting the above interval, we obtain the approximate $100(1-\alpha)\%$ confidence interval for the reliability as

$$\left(\frac{\hat{R}(t)}{\hat{R}(t) + (1 - \hat{R}(t))S(t)}, \frac{\hat{R}(t)}{\hat{R}(t) + (1 - \hat{R}(t))/S(t)}\right),$$
(3.47)

where

$$S(t) = \exp\left\{z_{1-\frac{\alpha}{2}}\frac{\widehat{se}(\hat{R}(t))}{\hat{R}(t)(1-\hat{R}(t))}\right\}.$$
(3.48)

Similarly, the distribution of $\widehat{E(T)}$ may be highly shewed when sample size is small or moderate. Pulcini [52] mentioned that, for constructing confidence intervals for mean lifetime, the normal approximation for the distribution of $\log(\widehat{E(T)})$ should be used, rather than for the distribution of $\widehat{E(T)}$, and that it would avoid the lower bound being negative. Bishop *et al.* [5] also pointed out that if $\log(\widehat{E(T)})$ is asymptotically normally distributed, the distribution of

$$g_2 = \frac{\log(\widehat{E(T)}) - \log(E(T))}{\widehat{se}(\widehat{E(T)})/\widehat{E(T)}}$$
(3.49)

is then asymptotically standard normal. This yields an approximate $100(1 - \alpha)\%$ confidence interval for the mean lifetime as

$$\left(\widehat{E(T)}\exp\left(\frac{-z_{1-\frac{\alpha}{2}}\widehat{se}(\widehat{E(T)})}{\widehat{E(T)}}\right), \widehat{E(T)}\exp\left(\frac{z_{1-\frac{\alpha}{2}}\widehat{se}(\widehat{E(T)})}{\widehat{E(T)}}\right)\right).$$
(3.50)

With the assumption of exponential distribution and the log-linear link function, it is observed that

$$\log(\widehat{E(T)}) = -\sum_{j=0}^{J} \hat{a}_j x_{0j}, \qquad (3.51)$$

the sum of the MLEs of the model parameters in a linear form. Due to the asymptotic normality of the MLEs \hat{a}_j , we have $\log(\widehat{E(T)})$ also to be approximately normally distributed and so confidence intervals for the mean lifetime of devices based on the log-transformation would work well in terms of coverage probability.

3.5 Illustrative Example

In this section, we present a numerical example to illustrate all the methods of inference developed in the preceding sections. Consider the following scenario wherein 10 one-shot devices were subjected to 2 types of stress factors, each of which were at 2 different levels, and observed at 3 pre-specified inspection times. Therefore, there were 120 devices in total placed under 12 different conditions in the experiment. The data collected are summarized in Table 3.1.

The point and the interval estimation of the parameters of interest were then found by the methods developed in the preceding sections, and the obtained results are presented in Table 3.2. Table 3.2 shows that, except in the case of estimation of the mean lifetime, the EM estimates are quite close to the jackknifed estimates of the parameters, and the bootstrap method yields confidence intervals that are quite similar to the asymptotic confidence intervals as well as the jackknifed confidence intervals. Moreover, the confidence intervals from the transformation approach (logit for reliability and log for mean lifetime) are closer to the bootstrap confidence intervals than the asymptotic confidence intervals. Based on the model, the expected number of failures for each testing condition can be computed, which are listed in Table 3.1. To test the suitability of the exponential model for the observed data, we employed the bootstrap method to approximate the *p*-value of the distance-based test statistic. For this purpose, we generated 100,000 samples from the exponential model with the estimates (-6.4573, 0.0340, (0.0301) as the true parameters. The test statistic for each bootstrap sample is then computed. Finally, the proportion of the test statistics from the bootstrap samples exceeding the value for the above data is calculated as the approximate *p*-value. The distance-based test statistic $K = \max_i |n_i - \hat{n}_i|$, which measures the absolute distance between the observed and the expected number of failures at each testing condition and takes the maximum, is used as a discrepancy measure

Testing $condition(i)$	(IT_i)	(K_i)	(n_i)	(\hat{n}_i)	(x_{i1})	(x_{i2})
1	2	10	0	1.54	55	70
2	2	10	4	3.38	55	100
3	2	10	4	3.71	85	70
4	2	10	7	6.81	85	100
5	5	10	4	3.42	55	70
6	5	10	7	6.44	55	100
7	5	10	8	6.86	85	70
8	5	10	8	9.43	85	100
9	8	10	3	4.88	55	70
10	8	10	9	8.08	55	100
11	8	10	9	8.43	85	70
12	8	10	10	9.90	85	100

Table 3.1: Data on 120 one-shot devices subjected to 2 types of stress factors and observed at 3 inspection times.

Table 3.2: Point estimates and 95% confidence intervals for the reliabilities at time $t = \{10, 30, 60\}$ and the mean lifetime at normal conditions $\boldsymbol{x}_0 = (25, 35)$ for the data presented in Table 3.1.

Estimates	a_0	a_1	a_2	
EM	-6.4573	0.0340	0.0301	
JK	-6.2788	0.0327	0.0289	
	R(10)	R(30)	R(60)	E(T)
EM	0.9001	0.7293	0.5319	95.034
JK	0.9110	0.7446	0.5295	59.637
95% CIs	<i>a</i> ₀	a_1	a_2	
FI	(-8.510, -4.405)	(0.016, 0.052)	(0.012, 0.048)	
JK	(-8.555, -4.003)	(0.014, 0.052)	(0.010, 0.048)	
ВТ	(-9.019, -4.423)	(0.017, 0.055)	(0.013, 0.051)	
	R(10)	R(30)	R(60)	E(T)
FI	(0.778, 1)	(0.433, 1)	(0.100, 0.964)	(0, 217.38)
JK	(0.785,1)	(0.436, 1)	(0.073, 0.986)	(0, 202.88)
BT	(0.694, 0.979)	(0.334, 0.938)	(0.112, 0.880)	(27.37, 468.04)
LOGIT	(0.699, 0.972)	(0.375, 0.924)	(0.167, 0.866)	-
LOG	-	-	-	(26.23, 344.35)

for evaluating the fit of the assumed exponential model to the observed data. We find in this case K = 1.8779 and the corresponding approximate *p*-value to be 0.8465, which gives very strong evidence towards the exponential distributional assumption made in our analysis.

3.6 Simulation Results

We carried out Monte Carlo simulation studies of size 10,000 to examine the proposed methods of point estimation as well as variance estimation of the estimates, and also for the estimates of the reliability at different mission times and the mean lifetime, for different sample sizes and levels of reliability. We considered the devices to have exponential lifetimes, subjected to two types of stress factors at two different conditions each, and tested at three different inspection times. Thus, in total, there were 12 testing conditions in the experiment. The stresses were taken as $x_1 = \{55, 80\}$ and $x_2 = \{70, 100\}$, and the inspection times were taken as $IT = \{2, 5, 8\}$. To estimate the true coefficients of the level of reliability taken to be $a_0 = \{-6.5, -6, -5.5\}$ and the stress factors $(a_1, a_2) = (0.03, 0.03)$, from different sample sizes $K = \{10, 50, 100, 200\}$, the number of failures in each testing condition was generated from the Binomial $(K, 1 - \exp(-IT \exp(\sum_{j=0}^{2} a_j x_j)))$ distribution. Then, the estimation of the reliability at normal stress level of $\boldsymbol{x}_0 = (25, 35)$ and at mission times $t = \{10, 30, 60\}$ were studied. In Tables 3.3 to 3.20, we present the simulated values of bias, mean square errors (MSE), coverage probabilities (CP) of 95% confidence intervals of some parameters of interest, and the corresponding average widths (AW). For the parametric bootstrap confidence intervals, we used the bootstrap sample size B as 999.

From these tables, we observe that as the sample size gets larger, the bias and the MSE both decrease and the confidence intervals become narrower and the CP also gets closer to the nominal level. Moreover, precise point estimation of the parameters and the mean lifetime are achieved by using the jackknife technique. On the other hand, the jackknife technique provides accurate point estimates of the reliability for short period, but not for long period in low reliability (see Table 3.4). In the variance estimation, all the proposed methods produce proper confidence intervals for the model parameters with desired CP even in the case of small sample sizes. Moreover, in terms of AW, the asymptotic confidence interval by the logittransformation produces the narrowest width with satisfactory CP in most of the considered situations.

When the sample size is not large, the CP of the asymptotic confidence intervals and the jackknife confidence intervals for the reliability at mission time and the mean lifetime are deflated. The reason is that both these confidence intervals require normal distribution, but due to the highly non-linear form of the reliability function and the mean lifetime, these distributions, however, are skewed in the case of small sample sizes. Figure 3.1 shows the skewed nature of the distribution of the MLEs of the reliability at mission time 10 obtained by one-step Newton-Raphson method for a sample of size 10, and the very nearly symmetric shape when the sample size is increased to 50. Hence, the bootstrap method is an appropriate

K = 10K = 50K = 100Bias in case of low reliability K = 200 $a_0 = -5.5$ EM estimate -1.469e-1-3.723e-2-8.588e-3-4.201e-3 Jackknife estimate 6.945e-2-6.274e-36.346e-33.166e-3 $a_1 = 0.03$ EM estimate 1.115e-32.565e-48.437e-53.875e-5Jackknife estimate -5.234e-4-3.183e-5-1.854e-51.616e-5 $a_2 = 0.03$ 1.241e-3EM estimate 3.014e-46.464e-53.977e-5Jackknife estimate -4.646e-4-5.146e-5-1.757e-56.088e-5R(10) = 0.781EM estimate -1.834e-2-2.672e-3-2.348e-3-1.279e-3Jackknife estimate -4.346e-37.206e-4-6.218e-4 -4.107e-4 R(30) = 0.476EM estimate 6.565e-33.028e-3-4.677e-4 -3.599e-4Jackknife estimate -8.765e-31.277e-3-1.283e-3-7.415e-4 R(60) = 0.227EM estimate 5.028e-21.358e-24.851e-42.429e-3 Jackknife estimate 2.858e-3-1.250e-31.789e-3-6.541e-4 E(T) = 40EM estimate 1.809e12.811e01.079e05.351e-1Jackknife estimate -1.028e1-8.362e-3 -2.145e-1-9.198e-2

Table 3.3: Values of bias of the estimates of the parameters a_0 , a_1 , a_2 , R(10), R(30), R(60), and E(T) for various sample sizes in the case of low reliability.

Table 3.4: Values of mean square errors of the estimates of the parameters a_0 , a_1 , a_2 , R(10), R(30), R(60), and E(T) for various sample sizes in the case of low reliability.

MSE in case	of low reliability	K = 10	K = 50	K = 100	K = 200
$a_0 = -5.5$	EM estimate	1.427e0	e0 2.509e-1 1.185e-1 6.091	2.509e-1 1.185e-1	
	Jackknife estimate	1.252e0	2.446e-1	1.174e-1	6.062e-2
$a_1 = 0.03$	EM estimate	1.054e-4	1.798e-5	8.868e-6	4.415e-6
	Jackknife estimate	9.159e-5	1.756e-5	8.778e-6	4.393e-6
$a_2 = 0.03$	EM estimate	1.043e-4	1.794e-5	8.605e-6	4.460e-6
	Jackknife estimate	8.995e-5	1.749e-5	8.522e-6	4.438e-6
R(10) = 0.781	EM estimate	1.867e-2	3.437e-3	1.649e-3	8.496e-4
	Jackknife estimate	1.901e-2	3.409e-3	1.638e-3	8.465e-4
R(30) = 0.476	EM estimate	4.403e-2	1.070e-2	5.297e-3	2.772e-3
	Jackknife estimate	5.549e-2	1.129e-2	5.447e-3	2.812e-3
R(60) = 0.227	EM estimate	4.436e-2	1.002e-2	4.834e-3	2.509e-3
	Jackknife estimate	5.098e-2	1.048e-2	4.976e-3	2.547e-3
E(T) = 40	EM estimate	5.313e3	1.981e2	8.028e1	3.879e1
	Jackknife estimate	7.613e2	1.583e2	7.285e1	3.700e1

Table 3.5: Values of coverage probabilities of 95% confidence intervals for the parameters a_0 , a_1 , and a_2 for various sample sizes in the case of low reliability.

CP of 95%	CI in case of low reliability	K = 10	K = 50	K = 100	K = 200
$a_0 = -5.5$	Asymptotic CI	0.954	0.948	0.954	0.949
	Jackknife CI	0.959	0.951	0.956	0.949
	Bootstrap CI	0.937	0.944	0.952	0.950
$a_1 = 0.03$	Asymptotic CI	0.950	0.949	0.948	0.948
	Jackknife CI	0.965	0.955	0.949	0.949
	Bootstrap CI	0.933	0.947	0.948	0.947
$a_2 = 0.03$	Asymptotic CI	0.952	0.949	0.953	0.950
	Jackknife CI	0.967	0.953	0.954	0.950
	Bootstrap CI	0.935	0.946	0.951	0.949

Table 3.6: Values of coverage probabilities of 95% confidence intervals for the parameters R(10), R(30), R(60), and E(T) for various sample sizes in the case of low reliability.

CP of 95% CI i	in case of low reliability	K = 10	K = 50	K = 100	K = 200
R(10) = 0.781	Asymptotic CI	0.886	0.928	0.946	0.945
	Jackknife CI	0.871	0.925	0.945	0.945
	Bootstrap CI	0.939	0.945	0.952	0.949
	CI by LOGIT	0.967	0.950	0.954	0.950
R(30) = 0.476	Asymptotic CI	0.856	0.923	0.945	0.943
	Jackknife CI	0.834	0.919	0.943	0.943
	Bootstrap CI	0.939	0.945	0.952	0.949
	CI by LOGIT	0.968	0.953	0.956	0.951
R(60) = 0.227	Asymptotic CI	0.825	0.916	0.935	0.939
	Jackknife CI	0.797	0.904	0.927	0.935
	Bootstrap CI	0.939	0.945	0.952	0.949
	CI by LOGIT	0.951	0.949	0.956	0.951
E(T) = 40	Asymptotic CI	0.905	0.941	0.949	0.945
	Jackknife CI	0.877	0.929	0.940	0.942
	Bootstrap CI	0.939	0.945	0.952	0.949
	CI by LOG	0.952	0.948	0.954	0.949

Table 3.7: Values of average widths of 95% confidence intervals for the parameters a_0 , a_1 , and a_2 for various sample sizes in the case of low reliability.

AW of 95%	CI in case of low reliability	K = 10	K = 50	K = 100	K = 200
$a_0 = -5.5$	Asymptotic CI	4.493e0	1.935e0	1.362e0	9.616e-1
	Jackknife CI	4.830e0	1.958e0	1.370e0	9.648e-1
	Bootstrap CI	5.051e0	1.971e0	1.377e0	9.679e-1
$a_1 = 0.03$	Asymptotic CI	3.616e-2	1.585e-2	1.118e-2	7.896e-3
	Jackknife CI	3.784e-2	1.598e-2	1.122e-2	7.912e-3
	Bootstrap CI	3.846e-2	1.605e-2	1.127e-2	7.943e-3
$a_2 = 0.03$	Asymptotic CI	3.616e-2	1.585e-2	1.118e-2	7.896e-3
	Jackknife CI	3.784e-2	1.598e-2	1.122e-2	7.912e-3
	Bootstrap CI	3.844e-2	1.604e-2	1.127e-2	7.939e-3

Table 3.8: Values of average widths of 95% confidence intervals for the parameters R(10), R(30), R(60), and E(T) for various sample sizes in the case of low reliability.

AW of 95% CI	in case of low reliability	K = 10	K = 50	K = 100	K = 200
R(10) = 0.781	Asymptotic CI	4.771e-1	2.259e-1	1.603e-1	1.132e-1
	Jackknife CI	4.722e-1	2.268e-1	1.607e-1	1.134e-1
	Bootstrap CI	5.035e-1	2.276e-1	1.613e-1	1.138e-1
	CI by LOGIT	4.820e-1	2.248e-1	1.600e-1	1.132e-1
R(30) = 0.476	Asymptotic CI	7.243e-1	4.011e-1	2.883e-1	2.053e-1
	Jackknife CI	7.194e-1	4.043e-1	2.896e-1	2.058e-1
	Bootstrap CI	7.075e-1	3.891e-1	2.844e-1	2.042e-1
	CI by LOGIT	6.904e-1	3.817e-1	2.808e-1	2.025e-1
R(60) = 0.227	Asymptotic CI	6.145e-1	3.756e-1	2.731e-1	1.949e-1
	Jackknife CI	5.920e-1	3.747e-1	2.751e-1	1.957e-1
	Bootstrap CI	6.910e-1	3.689e-1	2.688e-1	1.935e-1
	CI by LOGIT	7.495e-1	3.777e-1	2.719e-1	1.945e-1
E(T) = 40	Asymptotic CI	1.526e2	5.152 e1	3.463e1	2.408e1
	Jackknife CI	1.371e2	5.269e1	3.500e1	2.421e1
	Bootstrap CI	4.918e2	5.790e1	3.667e1	2.479e1
	CI by LOG	5.479e7	5.463e1	3.564e1	2.442e1

Table 3.9: Values of bias of the estimates of the parameters a_0 , a_1 , a_2 , R(10), R(30), R(60), and E(T) for various sample sizes in the case of moderate reliability.

Bias in case of	moderate reliability	K = 10	K = 50	K = 100	K = 200
$a_0 = -6$	EM estimate	-1.403e-1	-2.550e-2	-1.246e-2	-5.887e-3
	Jackknife estimate	1.493e-2	-1.180e-3	6.198e-4	6.070e-4
$a_1 = 0.03$	EM estimate	1.019e-3	1.797e-4	7.488e-5	3.168e-5
	Jackknife estimate	-7.658e-5	-8.658e-6	-1.740e-5	-1.415e-5
$a_2 = 0.03$	EM estimate	9.713e-4	1.743e-4	9.718e-5	5.126e-5
	Jackknife estimate	-1.254e-4	-1.393e-5	4.841e-6	-5.421e-6
R(10) = 0.861	EM estimate	-1.495e-2	-3.020e-3	-1.570e-3	-7.920e-4
	Jackknife estimate	-9.955e-5	1.944e-6	-5.905e-5	-3.726e-5
R(30) = 0.638	EM estimate	-1.065e-2	-2.703e-3	-1.495e-3	-7.949e-4
	Jackknife estimate	-2.937e-3	-1.572e-4	-1.505e-4	-1.047e-4
R(60) = 0.407	EM estimate	1.981e-2	3.815e-3	1.802e-3	8.057e-4
	Jackknife estimate	1.044e-3	-9.586e-5	-1.249e-4	-1.513e-4
E(T) = 67	EM estimate	2.806e1	4.197e0	2.046e0	9.712e-1
	Jackknife estimate	-1.362e1	-3.183e-1	-7.204e-2	-5.567e-2

Table 3.10: Values of mean square errors of the estimates of the parameters a_0 , a_1 , a_2 , R(10), R(30), R(60), and E(T) for various sample sizes in the case of moderate reliability.

MSE in case of	moderate reliability	K = 10	K = 50	K = 100	K = 200
$a_0 = -6$	EM estimate	1.330e0	2.382e-1	1.192e-1	5.777e-2
	Jackknife estimate	1.198e0	2.339e-1	1.182e-1	5.752e-2
$a_1 = 0.03$	EM estimate	9.029e-5	1.638e-5	8.157e-6	4.064e-6
	Jackknife estimate	8.223e-5	1.612e-5	8.097e-6	4.050e-6
$a_2 = 0.03$	EM estimate	9.114e-5	1.650e-5	8.228e-6	4.044e-6
	Jackknife estimate	8.318e-5	1.624e-5	8.162e-6	4.027e-6
R(10) = 0.861	EM estimate	9.343e-3	1.574e-3	7.776e-4	3.752e-4
	Jackknife estimate	8.563e-3	1.530e-3	7.660e-4	3.723e-4
R(30) = 0.638	EM estimate	3.340e-3	7.263e-3	3.710e-3	1.820e-3
	Jackknife estimate	3.920e-3	7.461e-3	3.758e-3	1.831e-3
R(60) = 0.407	EM estimate	4.631e-2	1.128e-3	5.886e-3	2.920e-3
	Jackknife estimate	5.919e-2	1.204e-3	6.088e-3	2.970e-3
E(T) = 67	EM estimate	8.748e3	5.138e2	2.292e2	1.048e2
	Jackknife estimate	1.478e3	4.172e2	2.075e2	9.982e1

Table 3.11: Values of coverage probabilities of 95% confidence intervals for the parameters a_0 , a_1 , and a_2 for various sample sizes in the case of moderate reliability.

CP of 95%	CI in case of moderate reliability	K = 10	K = 50	K = 100	K = 200
$a_0 = -6$	Asymptotic CI	0.953	0.953	0.950	0.953
	Jackknife CI	0.961	0.954	0.952	0.953
	Bootstrap CI	0.939	0.950	0.949	0.953
$a_1 = 0.03$	Asymptotic CI	0.951	0.950	0.950	0.951
	Jackknife CI	0.965	0.952	0.952	0.952
	Bootstrap CI	0.939	0.948	0.949	0.952
$a_2 = 0.03$	Asymptotic CI	0.949	0.952	0.949	0.952
	Jackknife CI	0.965	0.955	0.949	0.953
	Bootstrap CI	0.939	0.950	0.948	0.952

Table 3.12: Values of coverage probabilities of 95% confidence intervals for the parameters R(10), R(30), R(60), and E(T) for various sample sizes in the case of moderate reliability.

CP of 95% CI i	in case of moderate reliability	K = 10	K = 50	K = 100	K = 200
R(10) = 0.861	Asymptotic CI	0.884	0.936	0.940	0.949
	Jackknife CI	0.861	0.929	0.938	0.948
	Bootstrap CI	0.938	0.952	0.948	0.952
	CI by LOGIT	0.961	0.953	0.951	0.953
R(30) = 0.638	Asymptotic CI	0.872	0.934	0.940	0.948
	Jackknife CI	0.849	0.930	0.937	0.948
	Bootstrap CI	0.938	0.952	0.948	0.952
	CI by LOGIT	0.969	0.956	0.953	0.953
R(60) = 0.407	Asymptotic CI	0.849	0.931	0.937	0.946
	Jackknife CI	0.817	0.923	0.934	0.945
	Bootstrap CI	0.938	0.952	0.948	0.952
	CI by LOGIT	0.959	0.958	0.952	0.953
E(T) = 67	Asymptotic CI	0.901	0.941	0.944	0.948
	Jackknife CI	0.878	0.925	0.938	0.947
	Bootstrap CI	0.938	0.952	0.948	0.952
	CI by LOG	0.952	0.953	0.950	0.953

Table 3.13: Values of average widths of 95% confidence intervals for the parameters a_0 , a_1 , and a_2 for various sample sizes in the case of moderate reliability.

AW of 95%	% CI in case of moderate reliability	K = 10	K = 50	K = 100	K = 200
$a_0 = -6$	Asymptotic CI	4.356e0	1.904e0	1.343e0	9.484e-1
	Jackknife CI	4.572e0	1.922e0	1.349e0	9.506e-1
	Bootstrap CI	4.672e0	1.931e0	1.355e0	9.539e-1
$a_1 = 0.03$	Asymptotic CI	3.616e-2	1.585e-2	1.118e-2	7.896e-3
	Jackknife CI	3.784e-2	1.598e-2	1.122e-2	7.912e-3
	Bootstrap CI	3.846e-2	1.605e-2	1.127e-2	7.943e-3
$a_2 = 0.03$	Asymptotic CI	3.616e-2	1.585e-2	1.118e-2	7.896e-3
	Jackknife CI	3.784e-2	1.598e-2	1.122e-2	7.912e-3
	Bootstrap CI	3.844e-2	1.604e-2	1.127e-2	7.939e-3

Table 3.14: Values of average widths of 95% confidence intervals for the parameters R(10), R(30), R(60), and E(T) for various sample sizes in the case of moderate reliability.

AW of 95% CI in case of moderate reliability $% \left[$		K = 10	K = 50	K = 100	K = 200
R(10) = 0.861	Asymptotic CI	3.279e-1	1.532e-1	1.079e-1	7.607e-2
	Jackknife CI	3.150e-1	1.533e-1	1.079e-1	7.608e-2
	Bootstrap CI	3.668e-1	1.558e-1	1.090e-1	7.659e-2
	CI by LOGIT	3.623e-1	1.556e-1	1.088e-1	7.641e-2
R(30) = 0.638	Asymptotic CI	6.502e-1	3.321e-1	2.367e-1	1.680e-1
	Jackknife CI	6.403e-1	3.331e-1	2.370e-1	1.681e-1
	Bootstrap CI	6.284e-1	3.271e-1	2.354e-1	1.678e-1
	CI by LOGIT	6.071e-1	3.212e-1	2.327e-1	1.666e-1
R(60) = 0.407	Asymptotic CI	7.165e-1	4.146e-1	2.985e-1	2.131e-1
	Jackknife CI	6.998e-1	4.174e-1	2.995e-1	2.135e-1
	Bootstrap CI	7.007e-1	3.989e-1	2.931e-1	2.114e-1
	CI by LOGIT	7.081e-1	3.947e-1	2.906e-1	2.101e-1
E(T) = 67	Asymptotic CI	2.319e2	8.456e1	5.755e1	3.991e1
	Jackknife CI	2.139e2	8.632e1	5.809e1	4.009e1
	Bootstrap CI	6.523e2	9.442e1	6.086e1	4.107e1
	CI by LOG	3.867e2	8.968e1	5.925e1	4.049e1

Table 3.15: Values of bias of the estimates of the parameters a_0 , a_1 , a_2 , R(10), R(30), R(60), and E(T) for various sample sizes in the case of high reliability.

Bias in case of high reliability		K = 10	K = 50	K = 100	K = 200
$a_0 = -6.5$	EM estimate	-1.215e-1	-2.763e-2	-8.146e-3	-5.596e-3
	Jackknife estimate	2.402e-2	-1.898e-3	4.467e-3	6.704e-4
$a_1 = 0.03$	EM estimate	7.121e-4	1.579e-4	8.004e-5	4.076e-5
	Jackknife estimate	-2.157e-4	-5.869e-6	-2.611e-7	9.023e-7
$a_2 = 0.03$	EM estimate	8.365e-4	1.960e-4	3.324e-5	3.177e-5
	Jackknife estimate	-9.238e-5	3.218e-5	-4.693e-6	-8.069e-6
R(10) = 0.913	EM estimate	-1.363e-2	-2.517e-3	-1.519e-3	-6.808e-4
	Jackknife estimate	-1.393e-4	8.235e-5	-2.223e-4	-3.535e-5
R(30) = 0.761	EM estimate	-2.122e-2	-4.205e-3	-2.794e-3	-1.211e-3
	Jackknife estimate	-3.715e-3	9.456e-5	-5.896e-4	-9.588e-5
R(60) = 0.579	EM estimate	-9.257e-3	-1.857e-3	-2.016e-3	-7.324e-4
	Jackknife estimate	-6.246e-3	1.114e-4	-9.211e-4	-1.498e-4
E(T) = 110	EM estimate	5.051e1	7.922e0	3.435e0	1.817e0
	Jackknife estimate	-2.493e1	-3.709e-1	-4.325e-1	-6.370e-2

Table 3.16: Values of mean square errors of the estimates of the parameters a_0 , a_1 , a_2 , R(10), R(30), R(60), and E(T) for various sample sizes in the case of high reliability.

MSE in case of high reliability		K = 10	K = 50	K = 100	K = 200
$a_0 = -6.5$	EM estimate	1.412e0	2.640e-1	1.289e-1	6.461e-2
	Jackknife estimate	1.292e0	2.596e-1	1.280e-1	6.437e-2
$a_1 = 0.03$	EM estimate	9.594e-5	1.818e-5	8.854e-6	4.540e-6
	Jackknife estimate	8.915e-5	1.794e-5	8.795e-6	4.525e-6
$a_2 = 0.03$	EM estimate	9.579e-5	1.803e-5	8.833e-6	4.367e-6
	Jackknife estimate	8.874e-5	1.777e-5	8.782e-6	4.353e-6
R(10) = 0.913	EM estimate	4.984e-3	7.711e-4	3.690e-4	1.801e-4
	Jackknife estimate	4.134e-3	7.344e-4	3.592e-4	1.778e-4
R(30) = 0.761	EM estimate	2.304e-2	4.545e-3	2.238e-3	1.111e-3
	Jackknife estimate	2.437e-2	4.536e-3	2.230e-3	1.109e-3
R(60) = 0.579	EM estimate	4.151e-2	9.895e-3	5.013e-3	2.535e-3
	Jackknife estimate	5.174e-2	1.036e-2	5.125e-3	2.563e-3
E(T) = 110	EM estimate	3.218e4	1.644e3	6.995e2	3.341e2
	Jackknife estimate	4.280e3	1.312e3	6.303e2	3.171e2

Table 3.17: Values of coverage probabilities of 95% confidence intervals for the parameters a_0 , a_1 , and a_2 for various sample sizes in the case of high reliability.

CP of 95% CI in case of high reliability $% \left({{{\rm{D}}_{{\rm{B}}}} \right)$		K = 10	K = 50	K = 100	K = 200
$a_0 = -6.5$	Asymptotic CI	0.953	0.950	0.951	0.950
	Jackknife CI	0.964	0.954	0.953	0.949
	Bootstrap CI	0.940	0.947	0.952	0.948
$a_1 = 0.03$	Asymptotic CI	0.949	0.950	0.950	0.950
	Jackknife CI	0.962	0.953	0.953	0.951
	Bootstrap CI	0.941	0.947	0.949	0.948
$a_2 = 0.03$	Asymptotic CI	0.949	0.950	0.950	0.951
	Jackknife CI	0.962	0.954	0.951	0.951
	Bootstrap CI	0.942	0.946	0.949	0.951

Table 3.18: Values of coverage probabilities of 95% confidence intervals for the parameters R(10), R(30), R(60), and E(T) for various sample sizes in the case of high reliability.

CP of 95% CI in case of high reliability		K = 10	K = 50	K = 100	K = 200
R(10) = 0.913	Asymptotic CI	0.881	0.930	0.944	0.944
	Jackknife CI	0.853	0.921	0.939	0.943
	Bootstrap CI	0.941	0.947	0.952	0.948
	CI by LOGIT	0.957	0.951	0.953	0.948
R(30) = 0.761	Asymptotic CI	0.878	0.930	0.945	0.943
	Jackknife CI	0.851	0.924	0.941	0.943
	Bootstrap CI	0.941	0.947	0.952	0.948
	CI by LOGIT	0.967	0.953	0.954	0.949
R(60) = 0.579	Asymptotic CI	0.865	0.929	0.942	0.943
	Jackknife CI	0.836	0.922	0.940	0.942
	Bootstrap CI	0.941	0.947	0.952	0.948
	CI by LOGIT	0.966	0.953	0.954	0.949
E(T) = 110	Asymptotic CI	0.897	0.937	0.945	0.949
	Jackknife CI	0.871	0.922	0.937	0.944
	Bootstrap CI	0.941	0.947	0.952	0.948
	CI by LOG	0.953	0.949	0.952	0.948

Table 3.19: Values of average widths of 95% confidence intervals for the parameters a_0 , a_1 , and a_2 for various sample sizes in the case of high reliability.

AW of 95% CI in case of high reliability		K = 10	K = 50	K = 100	K = 200
$a_0 = -6.5$	Asymptotic CI	4.534e0	1.995e0	1.408e0	9.946e-1
	Jackknife CI	4.722e0	2.011e0	1.413e0	9.965e-1
	Bootstrap CI	4.782e0	2.020e0	1.419e0	1.000e0
$a_1 = 0.03$	Asymptotic CI	3.754e-2	1.655e-2	1.169e-2	8.258e-3
	Jackknife CI	3.894e-2	1.667e-2	1.172e-2	8.271e-3
	Bootstrap CI	3.929e-2	1.675e-2	1.177e-2	8.301e-3
$a_2 = 0.03$	Asymptotic CI	3.754e-2	1.655e-2	1.169e-2	8.257e-3
	Jackknife CI	3.894e-2	1.667e-2	1.172e-2	8.271e-3
	Bootstrap CI	3.928e-2	1.672e-2	1.177e-2	8.301e-3

Table 3.20: Values of average widths of 95% confidence intervals for the parameters R(10), R(30), R(60), and E(T) for various sample sizes in the case of high reliability.

AW of 95% CI in case of high reliability		K = 10	K = 50	K = 100	K = 200
R(10) = 0.913	Asymptotic CI	2.270e-1	1.058e-1	7.437e-2	5.225e-2
	Jackknife CI	2.132e-1	1.056e-1	7.430e-2	5.223e-2
	Bootstrap CI	2.768e-1	1.088e-1	7.557e-2	5.275e-2
	CI by LOGIT	2.797e-1	1.094e-1	7.570e-2	5.273e-2
R(30) = 0.761	Asymptotic CI	5.271e-1	2.593e-1	1.841e-1	1.300e-1
	Jackknife CI	5.100e-1	2.593e-1	1.840e-1	1.300e-1
	Bootstrap CI	5.438e-1	2.596e-1	1.846e-1	1.304e-1
	CI by LOGIT	5.267e-1	2.564e-1	1.832e-1	1.2974e-1
R(60) = 0.579	Asymptotic CI	7.089e-1	3.857e-1	2.767e-1	1.968e-1
	Jackknife CI	6.924e-1	3.868e-1	2.771e-1	1.969e-1
	Bootstrap CI	6.816e-1	3.755e-1	2.735e-1	1.959e-1
	CI by LOGIT	6.661e-1	3.682e-1	2.701e-1	1.943e-1
E(T) = 110	Asymptotic CI	4.049e2	1.496e2	1.010e2	7.021e1
	Jackknife CI	3.660e2	1.525e2	1.019e2	7.051e1
	Bootstrap CI	1.098e3	1.682e2	1.072e2	7.238e1
	CI by LOG	7.092e2	1.599e3	1.044e2	7.137e1



Figure 3.1: Histograms of the EM estimates of the reliability at mission time 10 obtained by one-step Newton-Raphson method for sample sizes of 10 and 50.

approach for constructing confidence intervals when the distribution of the pivoting quantity is skewed, as in the present situation. Moreover, as discussed before, since the logit-transformation of the reliability and the log-transformation of the mean lifetime are approximately normally distributed, these transformations result in confidence intervals with satisfactory CP and narrow AW, even when the sample size is small. Also, in our simulation study, we observed that the parametric bootstrap method is better than the transformation-based method for the case of low reliable devices, and this may be due to the fact that estimated standard deviation of the estimate is required for latter which tends to be not so precise. Tables 3.6, 3.12 and 3.18 show that the parametric bootstrap, employing the EM algorithm with one-step Newton-Raphson method, maintains acceptable CP even in the case of small sample sizes, and is therefore the method we would recommend for the purpose of interval estimation.

3.7 Concluding Remarks

We have developed here the EM algorithm with one-step Newton-Raphson method for finding the MLEs of the model parameters as well as of the reliability at a specific mission time and the mean lifetime, based on one-shot device testing data under the exponential distribution for lifetimes and consisting of many stress factors. Compared to the typical EM algorithm that finds the maximum likelihood estimates of parameters at each iteration, we find the one-step Newton-Raphson method to be quite efficient in this situation for finding the MLEs. For the point estimation of parameters, we have studied the jackknife technique as well. With this re-sampling technique, the bias and MSE of the estimates of the model parameters, the reliability at a specific time, and the mean lifetime are improved. However, the jackknife technique may provide an estimate that is outside the admissible range, especially when the sample size is small, and consequently suitable adjustment needs to be made on the estimate obtained by this jackknife method.

Furthermore, we have considered the confidence intervals constructed by using the observed Fisher information matrix, the jackknife technique, the parametric bootstrap method, and the transformation technique. In our simulation study, the asymptotic and jackknife confidence intervals for inference on the reliability and the mean lifetime are found to get deflated in the case of small sample sizes. But, the asymptotic method seems to produce suitable and short confidence intervals for the reliability at mission time and the mean lifetime when the sample size is large. We further observe that the parametric bootstrap method and the transformation technique are good alternative approaches for the construction of confidence intervals in the case of small sample sizes since the distributions of the MLEs of the reliability at mission time and the mean lifetime in this case are both not normally distributed and are in fact quite skewed. Since the parametric bootstrap confidence intervals may be centered wrongly in the case of small sample size due to the estimates being biased, one could also use the bias-corrected percentile method proposed by Efron [16], [17] which attempts to correct for the bias while constructing confidence intervals.

As mentioned earlier, it is reasonable to restrict the model parameters $a_j > 0$ in the framework of accelerated life-tests. But, the maximization in the M-step becomes more complicated in this case since we need to solve a restricted maximization problem. For finding the restricted estimates, Balakrishnan *et al.* [2], [3] developed the order-restricted maximum likelihood estimation of parameters for multiple step-stress models with exponentially distributed lifetimes under Type-I and Type-II censored sampling situations and also for sequential k-out-of-n systems.
Chapter 4

Weibull Lifetime Distribution with Multiple-Stress Model

4.1 Introduction

For analyzing reliability, there are many distributions to describe lifetime of devices, such as exponential, gamma and Weibull; see [26], [27]. Since Weibull distribution includes the exponential distribution as a special case and, in practice, it is quite widely used as a lifetime model in engineering, we consider in this Chapter Weibull distribution as the lifetime model for the devices. This would naturally generalize the results developed for the exponential case in the last two Chapters. In fact, the Weibull model is also used extensively in biomedical studies as a proportional hazards model for evaluating the effects of covariates on lifetimes, and in this setting usually the scale parameter varies with covariates but shape parameter remains unchanged over all covariates. Of course, under these assumptions, the Weibull distribution can be parametrized as a proportional hazards model meaning that the hazards of any two products stay in constant ratio. However, the shape parameter of the Weibull distribution may be different for different conditions in general, and so the assumption that the shape parameter does not depend on stress factors/covariates may often be violated in practice. Many examples show that both scale and shape parameters of the Weibull distribution vary with covariates (See [48], [60], [29]). Meeter and Meeker [39] presented more examples of the Weibull distribution with unequal shape parameters for modeling lifetimes of devices. For this reason, we develop here the EM algorithm for Weibull lifetime distribution when both scale and shape parameters vary over stress factors.

4.2 Model Description

We assume that lifetimes of the units, $\{t_{ik}, i = 1, 2, ..., I, k = 1, 2, ..., K_i\}$, have the Weibull distribution with pdf and cdf as

$$f_T(t,\alpha_i,\eta_i) = \frac{\eta_i t^{\eta_i - 1}}{\alpha_i^{\eta_i}} \exp\left(-\left(\frac{t}{\alpha_i}\right)^{\eta_i}\right), \qquad t > 0, \tag{4.1}$$

and

$$F_T(t, \alpha_i, \eta_i) = 1 - \exp\left(-\left(\frac{t}{\alpha_i}\right)^{\eta_i}\right), \qquad t > 0, \tag{4.2}$$

where $\alpha_i > 0$ and $\eta_i > 0$ are the scale and shape parameters, respectively.

It can be seen that the corresponding hazard function

$$h_T(t) = \frac{\eta_i t^{\eta_i - 1}}{\alpha_i^{\eta_i}} \tag{4.3}$$

is increasing when $\eta_i > 1$, implying that the unit suffers an increasing rate of failure as it ages, and decreasing when $\eta_i < 1$, meaning that the instantaneous failure probability falls as the unit gets older. In other words, better quality units remain and they have a lower failure rate. This latter case is not very likely in practice, but it may be applicable if we look at only the early part of the lifetimes of units. For $\eta_i = 1$, the Weibull distribution is equivalent to the exponential distribution and so the hazard rate is constant in this case. The Weibull distribution accommodates both increasing and decreasing failure rates simply based on the shape parameter. With a simple increasing or decreasing failure rate, it describes lifetime data in a flexible manner and for this reason is often used for modeling lifetime data. Figure 4.1 present a plot of the pdf, the reliability function, and the hazard function over time t for some choices of the shape parameter η (with scale parameter $\alpha = 1$). Moreover, the corresponding reliability function and the mean lifetime are

$$R(t) = \exp\left(-\left(\frac{t}{\alpha_i}\right)^{\eta_i}\right), \qquad t > 0, \tag{4.4}$$

and

$$E(T) = \alpha_i \Gamma\left(1 + \frac{1}{\eta_i}\right),\tag{4.5}$$

respectively. We assume that the parameters α_i and η_i both relate to the stress

Figure 4.1: Plots of the Weibull pdf, the reliability, and the hazard function for different choices of shape parameter.



levels in log-linear forms as

$$\alpha_i = \exp\left(\sum_{j=0}^J a_j x_{ij}\right) \tag{4.6}$$

and

$$\eta_i = \exp\left(\sum_{j=0}^J b_j x_{ij}\right). \tag{4.7}$$

Note that $x_{i0} \equiv 1$, for all *i*, corresponding to constant effects on the scale and shape parameters in the model.

Instead of working with Weibull lifetimes, it is often more convenient to work with the extreme value distribution for the log-lifetimes $w_{ik} = \log(t_{ik})$; see, for example, Meeter and Meeker [39] and Ng *et al.* [47]. Therefore, we consider here the extreme value distribution with the corresponding probability density function and distribution function as

$$f_W(w; \boldsymbol{\theta}) = \frac{1}{\sigma_i} \exp\left(\frac{w - \mu_i}{\sigma_i}\right) \exp\left(-\exp\left(\frac{w - \mu_i}{\sigma_i}\right)\right), -\infty < w < \infty, \quad (4.8)$$

and

$$F_W(w; \boldsymbol{\theta}) = 1 - \exp\left(-\exp\left(\frac{w - \mu_i}{\sigma_i}\right)\right), -\infty < w < \infty, \tag{4.9}$$

where $\theta = \{a_j, b_j, j = 0, 1, ..., J\}.$

The relationship between the two distributions is rather simple which can be utilized effectively for inferential purposes. Suppose a Weibull variable T has scale and shape parameters α_i and η_i . Then, $\log(T)$ has an extreme value distribution with location and shape parameters as

$$\mu_{i} = \log(\alpha_{i}) = \sum_{j=0}^{J} a_{j} x_{ij}$$
(4.10)

and

$$\sigma_i = \frac{1}{\eta_i} = \exp\left(-\sum_{j=0}^J b_j x_{ij}\right). \tag{4.11}$$

These two relations assist in the estimation of the parameters of the Weibull distribution once the estimates of the extreme value parameters have been determined.

For notational convenience, we denote $\mathbf{z} = \{W_i, K_i, n_i, i = 1, 2, ..., I\}$ for the observed data, where $W_i = \log(IT_i)$. Then, the likelihood function based on this observed data is given by

$$L(\boldsymbol{\theta}; \mathbf{z}) \propto \prod_{i=1}^{I} \left[F_W(W_i; \boldsymbol{\theta}) \right]^{n_i} \left[1 - F_W(W_i; \boldsymbol{\theta}) \right]^{K_i - n_i}, \qquad (4.12)$$

where $F_W(w; \boldsymbol{\theta})$ is in Eq. (4.9).

4.3 Point Estimation of Parameters of Interest

4.3.1 EM Algorithm Based on One-Step Newton-Raphson Method

Again, the EM algorithm is developed for the determination of the MLEs of the model parameters under the Weibull distribution. In the M-step, we need to maximize the log-likelihood function based on the relevant quantity updated from the E-step. Let

$$\xi_{ik} = \frac{w_{ik} - \mu_i}{\sigma_i}.\tag{4.13}$$

Then, the log-likelihood function based on the complete data can be expressed as

$$\ell_{c}(\theta) = \sum_{i=1}^{I} \sum_{k=1}^{K_{i}} \log(f_{W}(w_{ik};\theta))$$

=
$$\sum_{i=1}^{I} \sum_{k=1}^{K_{i}} \left(-\log(\sigma_{i}) + \xi_{ik} - e^{\xi_{ik}}\right) + \text{constant.}$$
(4.14)

For j = 0, 1, ..., J, taking the first-order derivatives of the above log-likelihood function with respect to the parameters a_j and b_j , we obtain the likelihood equations as

$$\frac{\partial \ell_c(\theta)}{\partial a_j} = \sum_{i=1}^{I} \sum_{k=1}^{K_i} \left(\frac{x_{ij}}{\sigma_i}\right) \left(-1 + e^{\xi_{ik}}\right), \qquad (4.15)$$

$$\frac{\partial \ell_c(\theta)}{\partial b_j} = \sum_{i=1}^{I} \sum_{k=1}^{K_i} x_{ij} \left(1 + \xi_{ik} - \xi_{ik} e^{\xi_{ik}} \right).$$
(4.16)

Here again, the MLEs of the model parameters, as solutions of the above likelihood equations, do not have explicit forms, and so an iterative numerical method such as the Newton-Raphson method need to be employed for this purpose. It requires the second-order derivatives of the log-likelihood function with respect to the parameters a_j and b_j , and these are as follows:

$$\frac{\partial^2 \ell_c(\theta)}{\partial a_p \partial a_q} = \sum_{i=1}^{I} \sum_{k=1}^{K_i} \left(-\frac{x_{ip} x_{iq}}{\sigma_i^2} e^{\xi_{ik}} \right), \tag{4.17}$$

$$\frac{\partial^2 \ell_c(\theta)}{\partial b_p \partial b_q} = \sum_{i=1}^{I} \sum_{k=1}^{K_i} x_{ip} x_{iq} \left(\xi_{ik} - \xi_{ik} e^{\xi_{ik}} - \xi_{ik}^2 e^{\xi_{ik}} \right), \tag{4.18}$$

$$\frac{\partial^2 \ell_c(\theta)}{\partial a_p \partial b_q} = \sum_{i=1}^I \sum_{k=1}^{K_i} \frac{x_{ip} x_{iq}}{\sigma_i} \left(-1 + e^{\xi_{ik}} + \xi_{ik} e^{\xi_{ik}} \right), \qquad (4.19)$$

for p = 0, 1, ..., J and q = 0, 1, ..., J. This would require four conditional expectations to be considered in the following E-step.

Given the model parameters, $\boldsymbol{\theta}$, let

$$\nu_i = \frac{W_i - \mu_i}{\sigma_i} \tag{4.20}$$

and

$$R_W(W_i; \boldsymbol{\theta}) = 1 - F_W(W_i; \boldsymbol{\theta}). \tag{4.21}$$

Then, the four required conditional expectations can be derived as follows:

$$E[\xi_i | \mathbf{z}, \boldsymbol{\theta}] = \frac{n_i}{K_i F_W(W_i; \boldsymbol{\theta})} \int_0^{e^{\nu_i}} \log(x) e^{-x} dx$$

+ $\frac{K_i - n_i}{K_i R_W(W_i; \boldsymbol{\theta})} \int_{e^{\nu_i}}^{\infty} \log(x) e^{-x} dx$
= $\frac{n_i}{K_i F_W(W_i; \boldsymbol{\theta})} \left(-\gamma - \nu_i e^{-e^{\nu_i}} - \int_{e^{\nu_i}}^{\infty} \frac{e^{-t}}{t} dt \right)$
+ $\frac{K_i - n_i}{K_i R_W(W_i; \boldsymbol{\theta})} \left(\nu_i e^{-e^{\nu_i}} + \int_{e^{\nu_i}}^{\infty} \frac{e^{-t}}{t} dt \right),$ (4.22)

$$E[e^{\xi_i}|\mathbf{z}, \boldsymbol{\theta}] = \frac{n_i}{K_i F_W(W_i; \boldsymbol{\theta})} \int_0^{e^{\nu_i}} x e^{-x} dx + \frac{K_i - n_i}{K_i R_W(W_i; \boldsymbol{\theta})} \int_{e^{\nu_i}}^{\infty} x e^{-x} dx$$
$$= \frac{n_i}{K_i F_W(W_i; \boldsymbol{\theta})} \left(1 - e^{-e^{\nu_i}} - e^{-e^{\nu_i} + \nu_i}\right)$$
$$+ \frac{K_i - n_i}{K_i R_W(W_i; \boldsymbol{\theta})} \left(e^{-e^{\nu_i}} + e^{-e^{\nu_i} + \nu_i}\right), \qquad (4.23)$$

$$E[\xi_{i}e^{\xi_{i}}|\mathbf{z},\boldsymbol{\theta}] = \frac{n_{i}}{K_{i}F_{W}(W_{i};\boldsymbol{\theta})} \int_{0}^{e^{\nu_{i}}} x \log(x)e^{-x}dx + \frac{K_{i}-n_{i}}{K_{i}R_{W}(W_{i};\boldsymbol{\theta})} \int_{e^{\nu_{i}}}^{\infty} x \log(x)e^{-x}dx, = \frac{n_{i}}{K_{i}F_{W}(W_{i};\boldsymbol{\theta})} \int_{0}^{e^{\nu_{i}}} x \log(x)d(-e^{-x}) + \frac{K_{i}-n_{i}}{K_{i}R_{W}(W_{i};\boldsymbol{\theta})} \int_{e^{\nu_{i}}}^{\infty} x \log(x)d(-e^{-x}) = \frac{n_{i}}{K_{i}F_{W}(W_{i};\boldsymbol{\theta})} \left(1-\gamma-e^{-e^{\nu_{i}}}-\nu_{i}e^{-e^{\nu_{i}}}-\nu_{i}e^{-e^{\nu_{i}}+\nu_{i}}-\int_{e^{\nu_{i}}}^{\infty} \frac{e^{-t}}{t}dt\right) + \frac{K_{i}-n_{i}}{K_{i}R_{W}(W_{i};\boldsymbol{\theta})} \left(e^{-e^{\nu_{i}}}+\nu_{i}e^{-e^{\nu_{i}}}+\nu_{i}e^{-e^{\nu_{i}+\nu_{i}}}+\int_{e^{\nu_{i}}}^{\infty} \frac{e^{-t}}{t}dt\right),$$

$$(4.24)$$

$$E[\xi_i^2 e^{\xi_i} | \mathbf{z}, \boldsymbol{\theta}] = \frac{n_i}{K_i F_W(W_i; \boldsymbol{\theta})} \int_0^{e^{\nu_i}} x(\log(x))^2 e^{-x} dx + \frac{K_i - n_i}{K_i R_W(W_i; \boldsymbol{\theta})} \int_{e^{\nu_i}}^{\infty} x(\log(x))^2 e^{-x} dx,$$
(4.25)

where $\int_x^{\infty} \frac{e^{-t}}{t} dt$ in Eqs. (4.22) and (4.24) is the exponential integral that can be readily computed by mathematical programs such as Matlab and Maple. For more details on the derivation of the last conditional expectation, we consider the integral $\int_0^{e^{\nu_i}} x \{\log(x)\}^2 e^{-x} dx$, and then have

$$\int_{0}^{e^{\nu_{i}}} x\{\log(x)\}^{2} e^{-x} dx = \int_{0}^{e^{\nu_{i}}} x\{\log(x)\}^{2} \sum_{m=0}^{\infty} \frac{(-x)^{m}}{m!} dx$$
$$= -\int_{0}^{e^{\nu_{i}}} \sum_{m=0}^{\infty} \frac{(-x)^{m+1} \{\log(x)\}^{2}}{m!} dx$$
$$= -\sum_{m=0}^{\infty} \frac{1}{m!} \int_{0}^{e^{\nu_{i}}} (-x)^{m+1} \{\log(x)\}^{2} dx$$
$$= \sum_{m=0}^{\infty} \frac{(-e^{\nu_{i}})^{m+2}}{m!(m+2)} \left\{ \nu_{i}^{2} - \frac{2\nu_{i}}{m+2} + \frac{2}{(m+2)^{2}} \right\}.$$
(4.26)

Moreover, it is simple to show that $\int_0^\infty x \{\log(x)\}^2 e^{-x} dx = \gamma^2 + \frac{1}{6}\pi^2 - 2\gamma$, where

 $\gamma \approx 0.577215665$ is Euler's constant. As a result, we have

$$\int_{e^{\nu_i}}^{\infty} x \{ \log(x) \}^2 e^{-x} dx = \gamma^2 + \frac{\pi^2}{6} - 2\gamma - \sum_{m=0}^{\infty} \frac{(-e^{\nu_i})^{m+2}}{m!(m+2)} \left\{ \nu_i^2 - \frac{2\nu_i}{m+2} + \frac{2}{(m+2)^2} \right\}.$$
 (4.27)

Consequently, we obtain

$$E[\xi_{i}^{2}e^{\xi_{i}}|\mathbf{z},\theta] = \frac{n_{i}}{K_{i}F_{W}(W_{i};\theta)} \left[\sum_{m=0}^{\infty} \frac{(-e^{\nu_{i}})^{m+2}}{m!(m+2)} \left\{\nu_{i}^{2} - \frac{2\nu_{i}}{m+2} + \frac{2}{(m+2)^{2}}\right\}\right] + \frac{K_{i} - n_{i}}{K_{i}R_{W}(W_{i};\theta)} \left[\gamma^{2} + \frac{\pi^{2}}{6} - 2\gamma - \sum_{m=0}^{\infty} \frac{(-e^{\nu_{i}})^{m+2}}{m!(m+2)} \left\{\nu_{i}^{2} - \frac{2\nu_{i}}{m+2} + \frac{2}{(m+2)^{2}}\right\}\right].$$

$$(4.28)$$

We can also develop inference on the reliability at time t, $R(t, \boldsymbol{x}_0; \hat{\boldsymbol{\theta}})$, as well as the mean lifetime of products, $E(T(\boldsymbol{x}_0; \boldsymbol{\hat{\theta}}))$, under normal operating conditions $\boldsymbol{x}_0 = \{x_{0j}, j = 0, 1, \dots, J\}$. Given \boldsymbol{x}_0 and the MLE, $\hat{\boldsymbol{\theta}}$, the corresponding estimators are simply given by

$$\hat{R}(t; \boldsymbol{x}_0) = \exp\left(-\exp\left(\frac{\log(t) - \hat{\mu}}{\hat{\sigma}}\right)\right)$$
(4.29)

and

$$\widehat{E(T)} = e^{\hat{\mu}} \Gamma \left(1 + \hat{\sigma}\right), \qquad (4.30)$$

where $\hat{\mu} = \exp\left(\sum_{i=0}^{J} \hat{a}_{i} x_{j}\right)$ and $\hat{\sigma} = \exp\left(-\sum_{j=0}^{J} \hat{b}_{j} x_{j}\right)$, and $\Gamma(\cdot)$ is the complete gamma function.

4.3.2 The Choice of Initial Guess of the EM Algorithm

As discussed by Lindsey and Ryan [33], the lifetimes can be first assumed to be distributed exponentially. In this way, we can assume that $\sigma_i = 1$ for all i = 1, 2, ..., I, compute μ_i by using the formula

$$\frac{n_i}{K_i} = 1 - e^{-e^{\nu_i}},\tag{4.31}$$

and then determine the parameters $\{a_0, a_1, \ldots, a_J\}$ through the least-squares method as described earlier. Moreover, unlike the typical Newton-Raphson method that finds MLEs in the M-step on each iteration, we adopt here the one-step Newton-Raphson method, as discussed by McLachlan and Krishnan [36].

Such an EM algorithm is given by the following iterative process:

- 1. Suppose $b_0^{(0)} = b_1^{(0)} = \cdots = b_J^{(0)} = 0$. Given \mathbf{z} , compute $\mu_i^{(0)}$ as well as $\sigma_i^{(0)}$, and then find $\{a_0^{(0)}, a_1^{(0)}, \dots, a_J^{(0)}\}$ through the least-squares method;
- 2. In the *m*-th iteration,
 - (a) in the E-step, compute the required conditional expectations $E[\xi_i | \mathbf{z}, \boldsymbol{\theta}^{(m)}], E[e^{\xi_i} | \mathbf{z}, \boldsymbol{\theta}^{(m)}], E[\xi_i e^{\xi_i} | \mathbf{z}, \boldsymbol{\theta}^{(m)}], \text{ and } E[\xi_i^2 e^{\xi_i} | \mathbf{z}, \boldsymbol{\theta}^{(m)}];$
 - (b) in the M-step, using the above conditional expectations, obtain the next iterate value of \(\mathcal{\mathcal{ heta}}^{(m+1)}\) by using the first step of the Newton-Raphson method.
- 3. Repeat Step 2 until convergence occurs to a desired level of accuracy, with the current $\boldsymbol{\theta}^{(m+1)}$ as the MLEs of model parameters, denoted by $\hat{\boldsymbol{\theta}}$.

4.4 Confidence Intervals for Parameters of Interest

Since there is no closed-form expression for the MLEs and that we cannot develop exact inference, we will describe here the asymptotic confidence intervals based on the observed Fisher information matrix, the parametric bootstrap confidence intervals, and also confidence intervals based on a transformation approach for some lifetime quantities of interest.

4.4.1 Use of Observed Fisher Information Matrix

Again, for the case of one-shot device testing data, due to the fact that all failure times are censored, the observed information matrix by Missing Information Principle is identical to the observed Fisher information matrix obtained from the log-likelihood function conditional on \mathbf{z} . The observed log-likelihood function is given by

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{I} [n_i \log (F_W(W_i; \boldsymbol{\theta})) + (K_i - n_i) \log (R_W(W_i; \boldsymbol{\theta}))] + \text{constant}$$

= $\sum_{i=1}^{I} [n_i \log(1 - e^{-e^{\nu_i}}) - (K_i - n_i)e^{\nu_i}] + \text{constant.}$ (4.32)

So, the second-order derivatives of the conditional distribution with respect to

the parameters a_j and b_j are derived as follows:

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial a_p \partial a_q} = -\sum_{i=1}^{I} \left(\frac{x_{ip} x_{iq}}{\sigma_i^2} \right) \times \left[n_i e^{-e^{\nu_i}} \left(\frac{-e^{\nu_i}}{1 - e^{-e^{\nu_i}}} + \left(\frac{-e^{\nu_i}}{1 - e^{-e^{\nu_i}}} \right)^2 \right) + (K_i - n_i) e^{\nu_i} \right], \quad (4.33)$$

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial b_p \partial b_q} = -\sum_{i=1}^{I} (x_{ip} x_{iq}) \\
\times \left[n_i e^{-e^{\nu_i}} \left(\frac{-\nu_i (\nu_i + 1) e^{\nu_i}}{1 - e^{-e^{\nu_i}}} + \left(\frac{-\nu_i e^{\nu_i}}{1 - e^{-e^{\nu_i}}} \right)^2 \right) + (K_i - n_i) (\nu_i + 1) \nu_i e^{\nu_i} \right],$$
(4.34)

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial a_p \partial b_q} = -\sum_{i=1}^{I} \left(\frac{x_{ip} x_{iq}}{\sigma_i} \right) \\
\times \left[n_i e^{-e^{\nu_i}} \left(\frac{-(\nu_i + 1) e^{\nu_i}}{1 - e^{-e^{\nu_i}}} + \nu_i \left(\frac{-e^{\nu_i}}{1 - e^{-e^{\nu_i}}} \right)^2 \right) + (K_i - n_i)(\nu_i + 1) e^{\nu_i} \right].$$
(4.35)

Then, the observed Fisher information matrix is

$$I_{obs} = \begin{pmatrix} -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial a_p \partial a_q} & -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial a_p \partial b_q} \\ -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial a_p \partial b_q} & -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial a_p \partial b_q} \end{pmatrix}.$$
(4.36)

The asymptotic variance-covariance matrix of the MLEs of the model parameters can then be obtained by inverting the above observed Fisher information matrix. In addition to the variances of the model parameters, the variance of the MLEs of the reliability at mission time t and the mean lifetime of units at normal operating conditions can also be computed by using the delta method, which requires the asymptotic variance-covariance matrix of the model parameters as well as the firstorder derivatives of the two estimates with respect to the model parameters. These are as follows:

$$\frac{\partial R(t; \boldsymbol{x}_0)}{\partial a_j} = \frac{x_j}{\hat{\sigma}} \exp\left(\frac{\log(t) - \hat{\mu}}{\hat{\sigma}}\right) \exp\left(-\exp\left(\frac{\log(t) - \hat{\mu}}{\hat{\sigma}}\right)\right),\tag{4.37}$$

$$\frac{\partial R(t; \boldsymbol{x}_0)}{\partial b_j} = -x_j \left(\frac{\log(t) - \hat{\mu}}{\hat{\sigma}}\right) \exp\left(\frac{\log(t) - \hat{\mu}}{\hat{\sigma}}\right) \exp\left(-\exp\left(\frac{\log(t) - \hat{\mu}}{\hat{\sigma}}\right)\right),$$
(4.38)

$$\frac{\partial E(T)}{\partial a_j} = x_j e^{\hat{\mu}} \Gamma \left(1 + \hat{\sigma}\right), \qquad (4.39)$$

$$\frac{\partial E(T)}{\partial b_j} = -x_j \hat{\sigma} e^{\hat{\mu}} \Psi(1+\hat{\sigma}) \Gamma\left(1+\hat{\sigma}\right), \qquad (4.40)$$

where $\Psi(\cdot)$ is the digamma function. By using all these expressions, the $100(1-\alpha)\%$ asymptotic confidence interval for any parameter of interest, ϕ , can be constructed as follows:

$$\left(\hat{\phi} - z_{1-\alpha/2}\widehat{s}\widehat{e}(\hat{\phi}), \hat{\phi} + z_{1-\alpha/2}\widehat{s}\widehat{e}(\hat{\phi})\right), \qquad (4.41)$$

where $\hat{\phi}$ is the MLE of ϕ , $\hat{se}(\hat{\phi})$ is the estimated standard error of $\hat{\phi}$, and $z_{1-\alpha/2}$ is the upper ($\alpha/2$)-th quantile of the standard normal distribution.

4.4.2 Use of Parametric Bootstrap Method

We now describe a procedure for constructing the parametric percentile bootstrap confidence interval for the parameter of interest, ϕ , which involves the following steps:

- 1. Obtain $\hat{\boldsymbol{\theta}}$ from the original sample \mathbf{z} ;
- 2. Simulate bootstrap samples, $\{n_i^*, i = 1, 2, ..., I\}$, from binomial distribution with (K_i, \hat{p}_i) , where $\hat{p}_i = 1 \exp\left(-\exp\left(\frac{\log(IT_i) \hat{\mu}}{\hat{\sigma}}\right)\right)$;
- 3. Obtain bootstrap estimate of ϕ , $\hat{\phi}^*$, based on the bootstrap samples through the EM algorithm with $\hat{\theta}$ as initial value;
- 4. Repeat Steps 2 and 3 B times to obtain a sample of bootstrap estimates, $\hat{\phi}^{(b)}$, for b = 1, 2, ..., B;
- 5. Arrange the bootstrap estimates in ascending order, denoted by $\hat{\phi}^{[b]}$, for b = 1, 2, ..., B.

Then, the $100(1-\alpha)\%$ percentile bootstrap confidence interval for ϕ is constructed as

$$\left(\hat{\phi}^{\left[\frac{\alpha}{2}(B+1)\right]}, \hat{\phi}^{\left[\left(1-\frac{\alpha}{2}\right)(B+1)\right]}\right). \tag{4.42}$$

4.4.3 Use of Transformation Method

Viveros and Balakrishnan [62] considered a transformation approach to construct confidence intervals for reliability. Even when the distribution of the estimate of reliability is skewed in the case of small samples, the bounds for the reliability always fall between 0 and 1 under this approach. Now, by employing a logit-transformation for the reliability, we have

$$\hat{g}_1 = \log\left(\frac{\hat{R}(t)}{1-\hat{R}(t)}\right) \tag{4.43}$$

to be asymptotically normally distributed with the corresponding standard deviation determined by the delta method, as

$$\widehat{se}(\widehat{g}) = \frac{\widehat{se}(\widehat{R}(t))}{\widehat{R}(t)(1 - \widehat{R}(t))},$$
(4.44)

where $\hat{se}(\hat{R}(t))$ is the estimated standard error of $\hat{R}(t)$. Therefore, we obtain an approximate $100(1-\alpha)\%$ confidence interval for the reliability R(t) to be

$$\left(\frac{\hat{R}(t)}{\hat{R}(t) + (1 - \hat{R}(t))S(t)}, \frac{\hat{R}(t)}{\hat{R}(t) + (1 - \hat{R}(t))/S(t)}\right),$$
(4.45)

where $S(t) = \exp\left\{z_{1-\frac{\alpha}{2}}\widehat{se}(\hat{g})\right\}$.

Similarly, Bishop *et al.* [5] mentioned a log-transformation approach for constructing confidence intervals for the mean lifetime, which avoids having negative lower bound for the mean lifetime. We, therefore, assume here that $\log(\widehat{E(T)})$ is asymptotically normally distributed with the corresponding standard deviation, by the delta method, as

$$\widehat{se}(\log(\widehat{E(T)})) = \frac{\widehat{se}(\widehat{E(T)})}{\widehat{E(T)}},$$
(4.46)

where $\widehat{se(E(T))}$ is the estimated standard error of $\widehat{E(T)}$. This results in an approximate $100(1-\alpha)\%$ confidence interval for the mean lifetime as

$$\left(\widehat{E(T)}\exp\left(\frac{-z_{1-\frac{\alpha}{2}}\widehat{se}(\widehat{E(T)})}{\widehat{E(T)}}\right), \widehat{E(T)}\exp\left(\frac{z_{1-\frac{\alpha}{2}}\widehat{se}(\widehat{E(T)})}{\widehat{E(T)}}\right)\right).$$
(4.47)

It should be noted that the required computational work for the estimated standard deviations of the reliability and the mean lifetime are exactly as presented earlier in Section 4.4.1.

4.5 Illustrative Example

In this section, we use mice tumor toxicological data to illustrate the proposed EM algorithm method. A real survival/sacrifice data obtained from the National Center for Toxicological Research is analyzed to study the relationships of dose of benzidine dihydrochloride, strain of offspring, and sex differences to time to appearance of tumors in mice. Many models have been studied in the literature for analyzing the survival/sacrifice data. Kodell and Nelson [29] specified an illnessdeath model under three independent Weibull distributions for time to occurrence of liver tumor with respect to different covariates. Finkelstein and Ryan [21] also suggested a proportional prevalence odds model to measure carcinogenic potency with benzidine dihydrochloride for liver tumor in terms of the log-odds of risk. Lindsey and Ryan [33] subsequently assumed that the death rate with liver tumor does not depend on the time to occurrence of the tumor in mice, and so presented a non-homogeneous Markov model with a multiplicative relationship between the hazard for death with and without the tumor, by assuming the baseline hazard model as a piecewise exponential distribution.

A survival/sacrifice data which involved 1816 mice, of which 553 had tumors, taken from the National Center for Toxicological Research, is presented in Table 4.1 and used to illustrate the inferential results developed in the preceding sections. These data have been considered earlier by Kodell and Nelson [29], Finkelstein and Ryan [21], and Lindsey and Ryan [33]. The original data were classified into 5 states and were reported by Kodell and Nelson [29]. Note that not all mice were sacrificed at pre-specified times since some died of tumors naturally before the sacrifice time. So, the time of natural death would also be treated as the time of sacrifice. We considered the mice sacrificed with tumors, died of tumors, and died of competing risks with liver tumors as those having tumors, while the mice sacrificed without tumors and died of competing risk without liver tumors as those not having tumors. Let a_1, a_2 and a_3 denote the parameters corresponding to the covariates of strain of offspring, gender, and concentration of the chemical of benzidine dihydrochloride in the scale parameter of the Weibull distribution, and b_1, b_2 and b_3 similarly for the shape parameter. We then computed, by using the EM algorithm, the MLEs and their standard errors, the 95% asymptotic and the parametric percentile bootstrap confidence intervals for all the model parameters, as well as the MLE of the mean time to occurrence of tumors for each group along with the corresponding standard error. These results are all presented in Tables 4.2 and 4.3.

Table 4.2 shows that the asymptotic confidence interval and the percentile bootstrap confidence interval for all model parameters are quite similar. Moreover, even though the proportional hazards model with Weibull distribution is often used in survival analysis, requiring same shape parameter for all groups, Table 4.2 shows

Table 4.1: The data from the tumorigenicity experiment of benzidine dihydrochloride in mice, giving the number of mice tested, K_i , number of mice having tumors, n_i , time to being sacrificed or natural death, IT_i , strain of offspring (F1 = 0, F2 = 1), x_{i1} , gender (F = 0, M = 1), x_{i2} , and concentration of the chemical (in ppm), x_{i3} .

K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}	K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}
48	1	9.27	0	0	60	1	1	16.70	0	0	60
24	0	9.37	0	0	60	24	0	9.27	0	0	120
24	1	13.97	0	0	60	24	0	9.37	0	0	120
24	2	14.03	0	0	60	24	5	13.97	0	0	120
36	18	18.63	0	0	60	23	9	14.03	0	0	120
1	1	16.00	0	0	60	26	25	18.67	0	0	120
1	1	16.70	0	0	60	1	1	12.83	0	0	120
1	1	18.10	0	0	60	1	1	13.47	0	0	120
1	1	18.30	0	0	60	1	1	14.60	0	0	120
1	1	18.47	0	0	60	1	1	14.80	0	0	120
1	0	12.57	0	0	60	1	1	15.17	0	0	120
1	0	14.43	0	0	60	1	1	16.53	0	0	120
1	0	14.57	0	0	60	1	1	16.57	0	0	120
1	0	17.17	0	0	60	1	1	16.90	0	0	120
1	0	18.83	0	0	60	1	1	17.43	0	0	120

K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}	K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}
2	2	17.47	0	0	120	1	1	13.20	0	0	200
1	1	17.70	0	0	120	1	1	13.53	0	0	200
1	1	17.90	0	0	120	1	1	14.03	0	0	200
1	1	18.40	0	0	120	1	1	14.23	0	0	200
1	0	9.17	0	0	120	1	1	14.60	0	0	200
1	0	12.47	0	0	120	1	1	14.70	0	0	200
1	0	14.70	0	0	120	2	2	14.73	0	0	200
1	0	15.40	0	0	120	1	1	14.83	0	0	200
1	0	17.30	0	0	120	1	1	14.93	0	0	200
1	0	18.13	0	0	120	1	1	15.43	0	0	200
1	1	13.50	0	0	120	2	2	15.53	0	0	200
1	1	13.96	0	0	120	1	1	16.33	0	0	200
1	1	14.33	0	0	120	1	1	16.63	0	0	200
1	1	15.13	0	0	120	1	1	16.80	0	0	200
1	1	15.40	0	0	120	1	1	16.93	0	0	200
47	4	9.33	0	0	200	1	1	18.13	0	0	200
45	38	14.00	0	0	200	1	0	7.93	0	0	200
4	4	18.67	0	0	200	1	0	8.20	0	0	200
1	1	11.10	0	0	200	1	1	18.67	0	0	200
1	1	12.13	0	0	200	24	16	9.33	0	0	400
1	1	12.17	0	0	200	10	9	14.00	0	0	400

K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}	K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}
1	1	9.87	0	0	400	1	1	14.40	0	0	400
1	1	11.03	0	0	400	1	1	14.60	0	0	400
1	1	11.20	0	0	400	1	1	14.70	0	0	400
1	1	12.93	0	0	400	1	1	15.27	0	0	400
1	1	13.17	0	0	400	1	1	18.00	0	0	400
1	1	14.27	0	0	400	1	1	10.63	0	0	400
1	1	11.80	0	0	400	1	1	11.30	0	0	400
1	1	11.93	0	0	400	1	1	11.93	0	0	400
3	3	12.50	0	0	400	1	1	13.10	0	0	400
1	1	12.97	0	0	400	48	0	9.27	0	1	120
2	2	13.07	0	0	400	44	7	14.00	0	1	120
1	1	13.17	0	0	400	22	7	18.73	0	1	120
1	1	13.20	0	0	400	20	4	19.30	0	1	120
2	2	13.60	0	0	400	1	1	13.27	0	1	120
1	1	13.63	0	0	400	1	1	13.70	0	1	120
2	2	13.67	0	0	400	1	1	17.40	0	1	120
1	1	13.83	0	0	400	1	1	19.07	0	1	120
2	2	13.87	0	0	400	1	0	9.83	0	1	120
1	1	14.03	0	0	400	1	0	10.23	0	1	120
1	1	14.10	0	0	400	1	0	11.63	0	1	120
2	2	14.30	0	0	400	1	0	12.13	0	1	120

K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}	K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}
1	0	12.77	0	1	120	1	1	14.73	0	1	400
1	0	17.00	0	1	120	1	1	17.43	0	1	400
47	3	9.33	0	1	120	1	1	17.53	0	1	400
32	5	14.00	0	1	120	1	1	18.50	0	1	400
19	8	18.67	0	1	120	1	1	18.67	0	1	400
1	1	11.97	0	1	120	1	0	10.07	0	1	400
1	1	12.90	0	1	120	1	0	11.23	0	1	400
1	1	15.13	0	1	120	1	1	17.10	0	1	400
1	1	15.63	0	1	120	23	0	9.30	1	0	60
1	0	4.63	0	1	120	47	0	9.37	1	0	60
1	0	6.60	0	1	120	24	5	14.03	1	0	60
1	0	7.57	0	1	120	24	5	14.07	1	0	60
1	0	8.83	0	1	120	17	8	18.63	1	0	60
1	0	9.80	0	1	120	18	7	18.70	1	0	60
1	0	13.77	0	1	200	1	1	15.63	1	0	60
1	0	17.83	0	1	200	1	1	16.20	1	0	60
1	0	17.92	0	1	200	1	1	16.53	1	0	60
24	0	9.37	0	1	400	1	1	17.60	1	0	60
22	11	14.00	0	1	400	1	1	17.90	1	0	60
15	11	18.70	0	1	400	1	1	18.50	1	0	60
1	1	7.870	0	1	400	1	1	18.53	1	0	60

K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}	K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}
1	0	7.53	1	0	60	1	1	16.53	1	0	120
1	0	8.63	1	0	60	2	2	16.57	1	0	120
1	0	13.73	1	0	60	1	1	16.83	1	0	120
1	0	14.93	1	0	60	1	1	17.87	1	0	120
1	0	15.53	1	0	60	2	2	18.00	1	0	120
1	0	15.63	1	0	60	1	1	18.37	1	0	120
1	0	17.13	1	0	60	1	0	3.07	1	0	120
24	2	9.27	1	0	120	1	0	5.20	1	0	120
22	0	9.37	1	0	120	1	0	5.60	1	0	120
41	15	14.00	1	0	120	1	0	5.63	1	0	120
21	20	18.70	1	0	120	1	0	5.67	1	0	120
1	1	11.63	1	0	120	1	0	8.87	1	0	120
1	1	13.70	1	0	120	1	0	9.27	1	0	120
1	1	14.00	1	0	120	1	0	9.80	1	0	120
1	1	14.53	1	0	120	1	0	11.40	1	0	120
1	1	14.66	1	0	120	1	0	11.77	1	0	120
1	1	14.83	1	0	120	1	0	13.07	1	0	120
1	1	15.13	1	0	120	1	0	15.13	1	0	120
1	1	15.27	1	0	120	1	0	15.40	1	0	120
1	1	16.13	1	0	120	1	1	11.87	1	0	120
1	1	16.37	1	0	120	1	1	13.57	1	0	120

K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}	K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}
1	1	13.60	1	0	120	2	2	15.43	1	0	200
1	1	18.33	1	0	120	1	1	16.07	1	0	200
23	2	9.30	1	0	200	1	1	16.93	1	0	200
24	2	9.33	1	0	200	1	1	17.07	1	0	200
18	9	14.00	1	0	200	1	1	17.83	1	0	200
17	14	14.03	1	0	200	1	1	18.00	1	0	200
3	3	18.70	1	0	200	1	0	6.700	1	0	200
1	1	11.90	1	0	200	1	0	9.17	1	0	200
1	1	12.10	1	0	200	1	0	13.93	1	0	200
1	1	12.30	1	0	200	1	0	17.07	1	0	200
1	1	13.03	1	0	200	1	0	18.27	1	0	200
1	1	13.07	1	0	200	1	1	9.33	1	0	200
2	2	13.30	1	0	200	1	1	11.43	1	0	200
1	1	13.33	1	0	200	1	1	11.83	1	0	200
1	1	13.60	1	0	200	1	1	12.23	1	0	200
2	2	13.73	1	0	200	22	8	9.33	1	0	400
2	2	13.77	1	0	200	14	13	14.00	1	0	400
1	1	13.97	1	0	200	1	1	18.63	1	0	400
1	1	14.20	1	0	200	2	2	11.93	1	0	400
1	1	14.87	1	0	200	2	2	12.20	1	0	400
2	2	15.27	1	0	200	1	1	12.70	1	0	400

K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}	K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}
1	1	12.80	1	0	400	1	1	9.80	1	0	400
3	3	12.83	1	0	400	1	1	11.43	1	0	400
2	2	13.13	1	0	400	1	1	12.33	1	0	400
1	1	13.70	1	0	400	1	1	12.83	1	0	400
1	1	14.23	1	0	400	1	1	13.93	1	0	400
1	1	14.77	1	0	400	24	0	9.27	1	1	60
1	1	14.87	1	0	400	23	0	9.30	1	1	60
1	1	15.10	1	0	400	21	0	9.37	1	1	60
1	1	15.40	1	0	400	44	3	14.00	1	1	60
1	1	16.80	1	0	400	18	2	18.67	1	1	60
1	1	16.97	1	0	400	20	2	18.70	1	1	60
1	1	17.47	1	0	400	1	1	16.77	1	1	60
1	1	18.23	1	0	400	1	1	17.47	1	1	60
1	0	3.90	1	0	400	1	1	18.30	1	1	60
1	0	6.70	1	0	400	1	0	1.93	1	1	60
1	0	7.93	1	0	400	1	0	3.07	1	1	60
1	0	9.80	1	0	400	1	0	4.77	1	1	60
1	0	12.83	1	0	400	2	0	6.23	1	1	60
1	0	13.80	1	0	400	1	0	6.53	1	1	60
1	0	13.93	1	0	400	1	0	7.00	1	1	60
1	0	16.43	1	0	400	1	0	8.23	1	1	60

K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}	K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}
1	0	9.73	1	1	60	47	0	9.30	1	1	200
1	0	10.00	1	1	60	37	6	14.00	1	1	200
1	0	13.03	1	1	60	17	7	18.67	1	1	200
1	0	16.23	1	1	60	1	1	13.60	1	1	200
1	0	16.42	1	1	60	1	0	1.90	1	1	200
1	0	16.67	1	1	60	2	0	3.67	1	1	200
24	0	9.30	1	1	120	1	0	7.00	1	1	200
22	0	9.37	1	1	120	4	0	7.74	1	1	200
37	8	14.00	1	1	120	1	0	10.73	1	1	200
21	7	18.63	1	1	120	1	0	12.03	1	1	200
18	5	18.70	1	1	120	1	0	13.10	1	1	200
1	1	9.57	1	1	120	1	0	13.97	1	1	200
1	1	14.43	1	1	120	1	0	14.60	1	1	200
1	1	17.87	1	1	120	1	0	16.23	1	1	200
1	1	18.03	1	1	120	1	0	16.97	1	1	200
1	0	5.13	1	1	120	1	1	15.43	1	1	200
1	0	7.13	1	1	120	24	1	9.30	1	1	400
1	0	8.07	1	1	120	21	4	14.00	1	1	400
1	0	12.23	1	1	120	12	6	18.67	1	1	400
1	0	17.60	1	1	120	1	1	11.90	1	1	400
1	0	18.23	1	1	120	1	1	14.77	1	1	400

K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}	K_i	n_i	IT_i	x_{i1}	x_{i2}	x_{i3}
1	1	17.00	1	1	400	1	0	9.07	1	1	400
1	1	17.63	1	1	400	1	0	10.93	1	1	400
1	1	17.93	1	1	400	1	0	12.83	1	1	400
1	1	18.50	1	1	400	1	0	14.00	1	1	400
1	0	6.43	1	1	400	1	1	12.37	1	1	400
1	0	8.17	1	1	400	1	1	17.47	1	1	400

that the shape parameter varies with the covariates. The same conclusion is also reached by the likelihood ratio test for equality of the shape parameters for all groups, that is, for testing the null hypothesis of $b_1 = b_2 = b_3 = 0$, is rejected with a *p*-value of 3.942×10^{-7} .

The estimates of the mean times to occurrence of tumors for each group in Table 4.3 are quite comparable to those obtained by Kodell and Nelson [29]. Also, Kodell and Nelson [29] did not report the estimates of the mean time to occurrence of tumors for the group of strain F1 mice with 60 ppm chemical, but it can be estimated by the model considered here through the log-linear link function. In addition, the reliability for each group at any time with any dose of the chemical can be estimated through the considered model as well.

For further analysis of the survival/sacrificed data, we may be interested in the effect of strain of offspring to the mean time to occurrence of tumors. Due to the sufficiently large sample size and the asymptotic property of MLEs, the MLEs of Table 4.2: The MLEs of the estimates of the model parameters, along with standard errors (within brackets) and the corresponding 95% asymptotic and parametric percentile bootstrap confidence intervals.

	scale parameter	shape parameter
Intercept	a_0	b_0
estimates (s.e.)	$2.9821 \ (0.0212)$	$1.9723 \ (0.1030)$
Asymptotic CI	(2.9406, 3.0236)	(1.7704, 2.1742)
Bootstrap CI	(2.9410, 3.0257)	(1.7758, 2.1882)
Strain of offspring	a_1	b_1
estimates (s.e.)	$0.0459\ (0.0224)$	-0.2102 (0.0894)
Asymptotic CI	(0.0021, 0.0897)	(-0.3855, -0.0349)
Bootstrap CI	(0.0006, 0.0875)	(-0.3935, -0.0293)
Gender	a_2	b_2
estimates (s.e.)	$0.5127 \ (0.0522)$	-0.4587(0.1151)
Asymptotic CI	(0.4103, 0.6150)	(-0.6844, -0.2331)
Bootstrap CI	(0.4238, 0.6357)	(-0.7041, -0.2498)
Concentration of chemical	<i>a</i> ₃	b_3
estimates (s.e.)	-0.0018 (0.0001)	-0.0014 (0.0004)
Asymptotic CI	(-0.0020, -0.0016)	(-0.0022, -0.0006)
Bootstrap CI	(-0.0021, -0.0016)	(-0.0021, -0.0005)

Table 4.3: The MLEs of the estimates of mean time to the occurrence of tumors, $\widehat{E(T)}$, along with standard errors (within brackets) and the corresponding 95% asymptotic and parametric percentile bootstrap confidence intervals.

Str.	Sex	Conc.	$\widehat{E(T)}$ in months	Asymptotic CI	Bootstrap CI
F1	F	60	16.4995 (0.2699)	(15.9705, 17.0284)	(15.9897, 17.0384)
		120	14.7288 (0.2066)	(14.3238, 15.1337)	(14.3300, 15.1552)
		200	$12.6572 \ (0.1966)$	(12.2719, 13.0426)	(12.2504, 13.0640)
		400	8.6610 (0.2800)	(8.1122, 9.2098)	(8.0944, 9.2110)
F1	М	60	26.8421 (1.3686)	(24.1596, 29.5247)	(24.5602, 30.3265)
		120	23.9562(1.1528)	(21.6968, 26.2156)	(22.0673, 26.8734)
		200	20.5908 (0.9406)	(18.7472, 22.4343)	(19.0756, 22.9599)
		400	14.1472 (0.6721)	(12.8298, 15.4645)	(13.0119, 15.8356)
F2	F	60	$17.0746\ (0.3349)$	(16.4182, 17.7310)	(16.4924, 17.7722)
		120	15.2379(0.2650)	(14.7185, 15.7574)	(14.7564, 15.7689)
		200	13.0919(0.2387)	(12.6242, 13.5597)	(12.6249, 13.5467)
		400	8.9640 (0.2948)	(8.3862, 9.5419)	(8.3003, 9.5259)
F2	М	60	27.7792 (1.4822)	(24.8742, 30.6842)	(25.4218, 31.4712)
		120	24.8081 (1.2593)	(22.3398, 27.2763)	(22.8085, 27.9218)
		200	21.3510 (1.0396)	(19.3134, 23.3886)	(19.6605, 23.9358)
		400	$14.7676 \ (0.7553)$	(13.2872, 16.2480)	(13.4943, 16.6077)

Table 4.4: Estimates of the mean times to the occurrence of tumors and the corresponding asymptotic variance-covariance matrix for the group of F1 strain of offspring.

	F1 strain of offspring $(n_1 = 828)$										
$\widehat{E(\boldsymbol{T}_1)} =$	16.499	26.842	14.729	23.956	12.657	20.591	8.661	14.147			
$\hat{\mathbf{\Sigma}}_1 =$	0.073	0.124	0.051	0.088	0.028	0.050	-0.009	-0.010			
	0.124	1.873	0.044	1.566	-0.040	1.229	-0.159	0.666			
	0.051	0.044	0.043	0.034	0.033	0.024	0.017	0.007			
	0.088	1.566	0.034	1.329	-0.021	1.067	-0.101	0.621			
	0.028	-0.040	0.033	-0.021	0.039	-0.003	0.043	0.023			
	0.050	1.229	0.024	1.067	-0.003	0.885	-0.042	0.566			
	-0.009	-0.159	0.017	-0.101	0.043	-0.042	0.078	0.046			
	-0.010	0.666	0.007	0.621	0.023	0.566	0.046	0.452			

mean times to occurrence of tumors for F1 and F2 strains of offspring have normal distribution with mean vector $\widehat{E(\mathbf{T}_1)}$ and variance-covariance matrix $\hat{\Sigma}_1$, and with mean vector $\widehat{E(\mathbf{T}_2)}$ and variance-covariance matrix $\hat{\Sigma}_2$, respectively. Therefore, a multivariate test for equality of mean vectors can be employed to test whether the mean time of F1 strain of offspring is equal to that of F2 strain of offspring. Note that the asymptotic variance-covariance matrices for both groups can be determined by the delta method from the asymptotic variance-covariance matrix of the MLEs of the model parameters. The mean times to occurrence of tumors

Table 4.5: Estimates of the mean times to the occurrence of tumors and the corresponding asymptotic variance-covariance matrix for the group of F2 strain of offspring.

	F2 strain of offspring $(n_2 = 988)$							
$\widehat{E(\boldsymbol{T}_2)} =$	17.075	27.779	15.238	24.808	13.092	21.351	8.964	14.768
$\hat{\mathbf{\Sigma}}_2 =$	0.112	0.209	0.084	0.161	0.053	0.109	0.004	0.027
	0.209	2.197	0.112	1.854	0.011	1.480	-0.137	0.860
	0.084	0.112	0.070	0.093	0.055	0.073	0.029	0.036
	0.161	1.854	0.093	1.586	0.023	1.291	-0.082	0.795
	0.053	0.011	0.055	0.023	0.057	0.034	0.055	0.045
	0.109	1.480	0.073	1.291	0.034	1.081	-0.024	0.718
	0.004	-0.137	0.029	-0.082	0.055	-0.024	0.087	0.055
	0.027	0.860	0.036	0.795	0.045	0.718	0.055	0.570

and the corresponding asymptotic variance-covariance matrices for both groups are presented in Tables 4.4 and 4.5. By Box's test, we conclude that asymptotic variance-covariance matrices for both groups are different. Since statistic $T^2 =$ $[(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)]' (\widehat{Cov(X_1)} + \widehat{Cov(X_2)})^{-1} [(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)]$ has an approximate χ_p^2 -distribution for large sample sizes (see Johnson and Wichern [28]), where p is the dimension size, we compute the T^2 -statistic to be 7.93 < 15.51 = $\chi_8^2(0.95)$. The null hypothesis that both groups are equal is not rejected, and we therefore conclude that the strain of offspring has no significant effect on the mean lifetime to occurrence of tumors, at 5% significance level.

4.6 Simulation Results

To evaluate the performance of the proposed EM algorithm for the MLEs, we carried out a Monte Carlo simulation study, for different levels of reliability and different sample sizes, to examine the bias, mean square error (MSE), coverage probability (CP), and average width (AW) of 95% confidence intervals. The devices were assumed to follow the Weibull distribution, and were on test under 3 different conditions with 1 stress factor at 3 levels. Then, all devices under each condition were tested at 3 different inspection times.

The sample size for each group (K_i) was taken to be 30, 50, and 100, corresponding to small, medium, and large sample sizes, respectively, and the model parameters were set as $(a_1, b_0, b_1) = (-0.05, -0.6, 0.03)$ while a_0 was chosen to be 4.9, 5.3, and 5.7 corresponding to devices with low, moderate, and high reliability, respectively. To prevent many zero-observations in test groups, the inspection times were not supposed to be the same for different levels of reliability. Specifically, the inspection times were set at $IT = \{5, 10, 15\}$ for the case of low reliability, $IT = \{8, 16, 24\}$ for the case of moderate reliability, and $IT = \{12, 24, 36\}$ for the case of high reliability. The results from the simulation study based on 10,000 Monte Carlo simulations are summarized in Tables 4.6 to 4.17. Due to heavy computation required for the construction of bootstrap confidence intervals, the simulation size for the bootstrap method was reduced to 1,000.

Table 4.6: Values of bias and mean square errors of the estimates of some parameters of interest for various sample sizes in case of low reliability.

Low reliability		K = 30	K = 50	K = 100	
	a_0	4.9	9.2809E-02	4.8126E-02	3.4581E-02
	a_1	-0.05	-1.8841E-03	-9.8457E-04	-7.1202E-04
	b_0	-0.6	4.9154E-03	1.2848E-02	-5.3638E-03
D.	b_1	0.03	1.7817E-04	-1.3976E-04	1.4977E-04
Bias	R(10; 25)	0.811	-1.8575E-04	4.5059E-04	-4.1252E-04
	R(20; 25)	0.627	-1.0987E-02	-6.3902E-03	-2.8449E-03
	R(30; 25)	0.473	-2.4833E-02	-1.5903E-02	-6.7668E-03
	E(T(25))	36.509	7.6808E+01	6.5957E + 00	2.8657E + 00
MSE	a_0	4.9	6.6536E-01	3.5485E-01	1.7081E-01
	a_1	-0.05	2.8205E-04	1.5106E-04	7.2815E-05
	b_0	-0.6	7.0445E-01	3.9971E-01	2.0042E-01
	b_1	0.03	3.3846E-04	1.9066E-04	9.4890E-05
	R(10; 25)	0.811	3.3348E-03	1.9555E-03	9.9009E-04
	R(20; 25)	0.627	5.3967E-03	2.9753E-03	1.3713E-03
	R(30; 25)	0.473	1.5372E-02	8.8881E-03	4.1433E-03
	E(T(25))	36.509	1.3109E+07	1.7006E + 03	1.6666E + 02

Table 4.7: Coverage probabilities and average widths of 95% confidence intervals for model parameters for various sample sizes in the case of low reliability.

Low reliability			K = 30	K = 50	K = 100
СР	$a_0 = 4.9$	Asymptotic CI	0.9272	0.9382	0.9449
		Bootstrap CI	0.9360	0.9410	0.9390
	$a_1 = -0.05$	Asymptotic CI	0.9322	0.9413	0.9464
		Bootstrap CI	0.9420	0.9410	0.9380
	$b_0 = -0.6$	Asymptotic CI	0.9464	0.9503	0.9491
		Bootstrap CI	0.9340	0.9420	0.9330
	$b_1 = 0.03$	Asymptotic CI	0.9436	0.9499	0.9501
		Bootstrap CI	0.9360	0.9370	0.9380
AW	$a_0 = 4.9$	Asymptotic CI	2.9747	2.2480	1.5752
		Bootstrap CI	3.2478	2.3746	1.5847
	$a_1 = -0.05$	Asymptotic CI	0.0614	0.0464	0.0325
		Bootstrap CI	0.0671	0.0490	0.0328
	$b_0 = -0.6$	Asymptotic CI	3.1589	2.4374	1.7207
		Bootstrap CI	3.2853	2.4677	1.6945
	$b_1 = 0.03$	Asymptotic CI	0.0689	0.0531	0.0374
		Bootstrap CI	0.0724	0.0541	0.0371

Low reliability			K = 30	K = 50	K = 100
СР		Asymptotic CI	0.9197	0.9343	0.9440
	R(10; 25) = 0.811	Bootstrap CI	0.9390	0.9380	0.9490
		CI by LOGIT	0.9423	0.9540	0.9494
	R(20;25) = 0.627	Asymptotic CI	0.9541	0.9519	0.9535
		Bootstrap CI	0.9490	0.9480	0.9630
		CI by LOGIT	0.9666	0.9578	0.9561
	R(30; 25) = 0.473	Asymptotic CI	0.9227	0.9315	0.9408
		Bootstrap CI	0.9420	0.9300	0.9480
		CI by LOGIT	0.9498	0.9487	0.9443
	E(T(25)) = 36.509	Asymptotic CI	0.8743	0.8945	0.9171
		Bootstrap CI	0.9320	0.9460	0.9380
		CI by LOG	0.9154	0.9331	0.9483

Table 4.8: Coverage probabilities of 95% confidence intervals for parameters of interest for various sample sizes in the case of low reliability.

Low reliability			K = 30	K = 50	K = 100
AW	D(10.05) = 0.011	Asymptotic CI	0.2189	0.1716	0.1223
	R(10; 25) = 0.811	Bootstrap CI	0.2166	0.1698	0.1210
		CI by LOGIT	0.2216	0.1729	0.1228
	R(20; 25) = 0.627	Asymptotic CI	0.2768	0.2098	0.1451
		Bootstrap CI	0.3002	0.2158	0.1450
		CI by LOGIT	0.2669	0.2048	0.1427
	R(10; 25) = 0.811 $R(20; 25) = 0.627$ $R(30; 25) = 0.473$ $E(T(25)) = 36.509$	Asymptotic CI	0.4422	0.3522	0.2463
		Bootstrap CI	0.4494	0.3497	0.2390
		CI by LOGIT	0.4246	0.3350	0.2375
	E(T(25)) = 36.509	Asymptotic CI	972.93	74.79	43.20
		Bootstrap CI	2.04e18	412.77	63.55
		CI by LOG	3.25e7	146.62	49.08

Table 4.9: Average widths of 95% confidence intervals for parameters of interest for various sample sizes in the case of low reliability.
Mode	rate reliabili	ty	K = 30	K = 50	K = 100
	a_0	5.3	8.1539E-02	5.5986E-02	2.5774E-02
	a_1	-0.05	-1.6980E-03	-1.1720E-03	-5.2485E-04
	b_0	-0.6	-2.3014E-03	-7.1066E-03	-3.3800E-03
D:	b_1	0.03	4.1892E-04	3.1498E-04	1.7543E-04
Bias	R(30; 25)	0.625	-7.9630E-03	-4.3641E-03	-1.8900E-03
	R(40; 25)	0.518	-1.7935E-02	-9.7984E-03	-4.5495E-03
	R(50; 25) 0.427		-2.2679E-02	-1.2818E-02	-6.2854E-03
	E(T(25))	54.465	6.0039E+01	9.4660E+00	3.3759E+00
	a_0	5.3	5.3693E-01	3.0023E-01	1.3661E-01
	a_1	-0.05	2.3002E-04	1.2927E-04	5.9231E-05
	b_0	-0.6	6.6179E-01	3.8185E-01	1.8045E-01
	b_1	0.03	3.1953E-04	1.8271E-04	8.6423E-05
MSE	R(30; 25)	0.625	4.3757E-03	2.5115E-03	1.2081E-03
	R(40; 25)	0.518	9.4680E-03	5.3203E-03	2.4480E-03
	R(50; 25)	0.427	1.5193E-02	9.0072E-03	4.2773E-03
	E(T(25))	54.465	9.5348E+06	3.5714E + 03	2.6959E + 02

Table 4.10: Values of bias and mean square errors of the estimates of some parameters of interest for various sample sizes in case of moderate reliability.

Mode	erate reliabili	ity	K = 30	K = 50	K = 100
	F 0	Asymptotic CI	0.9313	0.9351	0.9413
	$a_0 = 5.3$	Bootstrap CI	0.9370	0.9430	0.9340
	. 0.05	Asymptotic CI	0.9376	0.9394	0.9439
CD	$a_1 = -0.05$	Bootstrap CI	0.9370	0.9410	 K = 100 0.9413 0.9340 0.9439 0.9439 0.9496 0.9496 0.9496 0.9410 0.9485 0.9390 1.4188 1.4328 0.0295 0.0295 0.0298 1.6572 1.6363 0.0362 0.0359
CP	b 06	Asymptotic CI	0.9428	0.9466	0.9496
	$b_0 = -0.0$	Bootstrap CI	0.9390	0.9360	0.9410
	$b_1 = 0.03$	Asymptotic CI	0.9410	0.9450	0.9485
		Bootstrap CI	0.9340	0.9300	0.9390
	F 0	Asymptotic CI	2.6720	2.0397	1.4188
	$a_0 = 5.3$	Bootstrap CI	2.9562	2.1777	1.4328
	0.05	Asymptotic CI	0.0555	0.0424	0.0295
4337	$a_1 = -0.05$	Bootstrap CI	0.0613	0.0452	0.0298
AW		Asymptotic CI	3.0413	2.3500	1.6572
	$b_0 = -0.6$	Bootstrap CI	3.1658	2.3835	1.6363
		Asymptotic CI	0.0666	0.0514	0.0362
	$b_1 = 0.03$	Bootstrap CI	0.0699	0.0524	0.0359

Table 4.11: Coverage probabilities and average widths of 95% confidence intervals for model parameters for various sample sizes in the case of moderate reliability.

Moderate reliability			K = 30	K = 50	K = 100
	R(30;25) = 0.625	Asymptotic CI	0.9518	0.9529	0.9500
		Bootstrap CI	0.9390	0.9370	0.9450
		CI by LOGIT	0.9615	0.9579	0.9526
		Asymptotic CI	0.9459	0.9456	0.9473
	R(40; 25) = 0.518	Bootstrap CI	0.9350	0.9390	0.9390
CD		CI by LOGIT 0.964	0.9646	0.9566	0.9504
CP		Asymptotic CI	0.9125	0.9232	0.9393
	R(50;25) = 0.427	Bootstrap CI	0.9370	0.9430	0.9380
		CI by LOGIT	0.9427	0.9423	0.9463
		Asymptotic CI	0.8741	0.8960	0.9187
	E(T(25)) = 54.465	Bootstrap CI	0.9380	0.9440	0.9440
		CI by LOG	0.9153	0.9300	0.9459

Table 4.12: Coverage probabilities of 95% confidence intervals for parameters of interest for various sample sizes in the case of moderate reliability.

Mode	Moderate reliability			K = 50	K = 100
		Asymptotic CI	0.2553	0.1943	0.1357
	R(30;25) = 0.625	Bootstrap CI	0.2646	0.1964	0.1358
		CI by LOGIT	0.2481	0.1907	0.1340
	R(40;25) = 0.518	Asymptotic CI	0.3577	0.2734	0.1906
		Bootstrap CI	0.3662	0.2740	0.1872
4337		CI by LOGIT	0.3398	0.2629	0.1357 0.1358 0.1340 0.1906 0.1872 0.1859 0.2508 0.2417 0.2427 56.54 78.23
AW		Asymptotic CI	0.4366	0.3509	0.2508
	R(50;25) = 0.427	Bootstrap CI	0.4274	0.3411	0.2417
		CI by LOGIT	0.4291	0.3373	0.2427
		Asymptotic CI	704.37	102.90	56.54
	E(T(25)) = 54.465	Bootstrap CI	2.10e8	4.53e4	78.23
		CI by LOG	7.15e6	203.49	62.55

Table 4.13: Average widths of 95% confidence intervals for parameters of interest for various sample sizes in the case of moderate reliability.

s of interest for various sample sizes in case of high reliability.									
High	reliability		K = 30	K = 50	K = 100				
	a_0	5.7	8.2356E-02	4.8147E-02	2.1972E-02				
	a_1	-0.05	-1.7189E-03	-1.0034E-03	-4.5630E-04				
	b_0	-0.6	-1.8072E-02	-6.8206E-03	-3.4781E-04				
Biog	b_1	0.03	7.1480E-04	3.4118E-04	9.5830E-05				
Dias	R(60; 25)	0.516	-1.6584E-02	-9.9779E-03	-5.2695E-03				
	R(70; 25)	0.453	-1.9836E-02	-1.2367E-02	-6.6925E-03				
	R(80; 25)	0.397	-2.0227E-02	-1.3174E-02	-7.3891E-03				

Table 4.14: Values of bias and mean square errors of the estimates of some parameters of interest for various sample sizes in case of high reliability.

	R(80; 25)	0.397	-2.0227E-02	-1.3174E-02	-7.3891E-03
	E(T(25))	81.252	3.2269E+01	1.1906E + 01	4.8035E+00
	a_0	5.7	4.9017E-01	2.8095E-01	1.3493E-01
	a_1	-0.05	2.1097E-04	1.2117E-04	5.8483E-05
	b_0	-0.6	6.3601E-01	3.6596E-01	1.8169E-01
MOD	b_1	0.03	3.0820E-04	1.7567E-04	8.6444E-05
MSE	R(60; 25)	0.516	9.0062E-03	5.2017E-03	2.4766E-03
	R(70; 25)	0.453	1.2753E-02	7.5995E-03	3.6907E-03
	R(80; 25)	0.397	1.5717E-02	9.7851E-03	4.9210E-03
	E(T(25))	81.252	1.3631E + 05	3.3846E+03	5.6220E + 02

High	reliability		K = 30	K = 50	K = 100
	F 7	Asymptotic CI	0.9338	0.9358	0.9414
	$a_0 = 5.7$	Bootstrap CI	0.9350	0.9490	0.9460
	. 0.05	Asymptotic CI	0.9395	0.9385	0.9426
CD	$a_1 = -0.05$	Bootstrap CI	0.9420	0.9450	 K = 100 0.9414 0.9460 0.9426 0.9410 0.9465 0.9330 0.9465 0.9330 0.9370 1.4056 1.4056 1.4174 0.0292 0.0295 1.6526 1.6314 0.0361 0.0359
CP		Asymptotic CI	0.9486	0.9497	0.9465
	$b_0 = -0.0$	Bootstrap CI	0.9260	0.9440	 K = 100 0.9414 0.9460 0.9426 0.9410 0.9465 0.9330 0.9476 0.9370 1.4056 1.4174 0.0292 0.0295 1.6526 1.6314 0.0361 0.0359
	$b_1 = 0.03$	Asymptotic CI	0.9484	0.9503	0.9476
		Bootstrap CI	0.9300	0.9420	0.9370
	~	Asymptotic CI	2.6554	2.0168	1.4056
	$a_0 = 5.7$	Bootstrap CI	2.8973	2.1269	1.4174
	0.05	Asymptotic CI	0.0552	0.0419	0.0292
4337	$a_1 = -0.05$	Bootstrap CI	0.0602	0.0442	0.0295
AW		Asymptotic CI	3.0370	2.3427	1.6526
	$b_0 = -0.0$	Bootstrap CI	3.1585	2.3750	1.6314
		Asymptotic CI	0.0665	0.0512	0.0361
	$b_1 = 0.03$	Bootstrap CI	0.0698	0.0523	0.0359

Table 4.15: Coverage probabilities and average widths of 95% confidence intervals for model parameters for various sample sizes in the case of high reliability.

High	High reliability			K = 50	K = 100
		Asymptotic CI	0.9472	0.9502	0.9500
	R(60; 25) = 0.516	Bootstrap CI	0.9530	0.9420	0.9450
		CI by LOGIT	0.9665	0.9592	0.9531
	R(70; 25) = 0.453	Asymptotic CI	0.9251	0.9362	0.9434
		Bootstrap CI	0.9390	0.9370	0.9490
CD		CI by LOGIT	0.9511	0.9370 0.9519	0.9456
CP		Asymptotic CI	0.9074	0.9228	0.9358
	R(80;25) = 0.397	Bootstrap CI	0.9360	0.9400	0.9460
		CI by LOGIT	0.9386	0.9437	0.9414
		Asymptotic CI	0.8858	0.9024	0.9179
	E(T(25)) = 81.252	Bootstrap CI	0.9360	0.9430	0.9450
		CI by LOG	0.9225	0.9335	0.9465

Table 4.16: Coverage probabilities of 95% confidence intervals for parameters of interest for various sample sizes in the case of high reliability.

High	High reliability			K = 50	K = 100
		Asymptotic CI	0.3548	0.2729	0.1907
	R(60; 25) = 0.516	Bootstrap CI	0.3708	0.2771	0.1874
		CI by LOGIT	0.3370	0.2626	0.1860
		Asymptotic CI	0.4135	0.3276	0.2317
	R(70; 25) = 0.453	Bootstrap CI	0.4156	0.3248	0.2247
4117		CI by LOGIT	0.3979	0.3136	0.2247
AW		Asymptotic CI	0.4495	0.3694	0.2674
	R(80;25) = 0.397	Bootstrap CI	0.4383	0.3572	0.2555
		CI by LOGIT	0.4513	0.3579	0.2588
		Asymptotic CI	303.79	142.66	83.36
	E(T(25)) = 81.252	Bootstrap CI	4.14e5	568.08	113.44
		CI by LOG	2.83e4	208.44	91.96

Table 4.17: Average widths of 95% confidence intervals for parameters of interest for various sample sizes in the case of high reliability.

We observe first that the MSEs of the estimates of the model parameters as well as of the reliability at different times and of the mean lifetime all become smaller when sample size increases, and so does the bias too. Moreover, the MSE of the estimate of the reliability increases with time. We also observe that, while the point and interval estimates of the model parameters as well as the reliability by the EM algorithm are all satisfactory, even in the case of small samples sizes, the estimate of the mean lifetime is not satisfactory when the sample size is small, but does become better as the sample size increases. The percentile bootstrap method and the transformation approach work very well for confidence intervals for reliability prediction even for small sample sizes as they maintain the coverage probability at the nominal level of 95%. Also, the average width of the bootstrap interval is slightly larger than that of the asymptotic confidence interval, while the transformation approach yields the shortest confidence interval for reliability in most cases. We would therefore recommend the logit-transformation approach for constructing confidence intervals for the reliability. However, for the mean lifetime, the transformation approach does not work better than the bootstrap method in case of small samples. It is due to the form of the mean lifetime of the Weibull distribution, $\alpha \Gamma(1+1/\eta)$, which does not seem to result in a near normal distribution for its estimate. Finally, as we would expect, the coverage probability of asymptotic confidence intervals remain below the nominal level for small sample sizes, but get close to the nominal level as sample size becomes larger.

4.7 Concluding Remarks

The data considered here are subjected to both left- and right-censoring, and are quite common in reliability and survival analysis. The Weibull distribution is widely used for describing the lifetimes of devices and individuals. Unlike the typical Weibull analysis wherein the shape parameter in the model is assumed to be constant over all the stress levels/covariates, we have assumed here non-constant scale and shape parameters in the Weibull model. Since more parameters are introduced in the model, the model becomes more flexible, but the simulation study reveals that a larger sample size, however, is needed to obtain stable and accurate estimates. The simulation results also show that the EM algorithm developed here is quite satisfactory for the estimation of the model parameters as well as for the reliability, but not for the estimation of mean lifetime. For confidence intervals for reliability at a specific time, the bootstrap method and the transformation approach are observed to be more suitable than the asymptotic method, in case of small samples, but all these methods give quite similar results in case of large sample sizes.

Apart from the simulation study, a real data from a tumorigenicity experiment has been analyzed by the proposed method, and the confidence intervals for the model parameters obtained by the asymptotic method and the bootstrap method are quite similar. Moreover, the estimates of mean times to the occurrence of tumors for all the considered groups obtained by the proposed method are close to the results of Kodell and Nelson [29]. Based on the model, the estimated mean time and the corresponding standard errors are obtained. Furthermore, an effect of strain of offspring to the mean time to tumor occurrence has been evaluated by T^2 -test. Finally, the likelihood ratio test reveals that the shape parameters are indeed different over the gender, the strain of offspring, and the concentration of chemical.

Since the estimation of the mean lifetime of Weibull distribution is of great interest for reliability engineers, it is natural to seek another method that yields a better confidence interval in case of small sample sizes. As mentioned before, the mean lifetime of the Weibull distribution is $\alpha\Gamma(1 + 1/\eta)$. Viveros and Balakrishnan [63] pointed out that the lack of a pivotal quantity for the mean lifetime of the Weibull distribution makes it impossible to derive exact conditional (or unconditional) confidence intervals. Instead of using logarithm of the mean lifetime, we may consider some other transformations which produce near normality for the distribution of the quantity. Work in this direction will be of great interest.

Chapter 5

Optimal Accelerated Life-Test Plans for One-Shot Device Testing

5.1 Introduction

Traditional plans consist of equally-spaced stress levels and inspection time, each with the same number of test units, which usually yields less accurate estimates of a lifetime parameter of interest. Within a pre-fixed budget, optimal test plans that minimize the asymptotic variance of the estimate of a parameter of interest are often constructed to collect information efficiently from ALTs. Due to their practical importance, much work has been done on optimal test plans. Nelson [44], [45] has provided a list of publications that have dealt with optimal accelerated life-test plans. Sohn [56] has developed the optimal ALTs for intermittent destructive inspection with logistic lifetime distribution. Tseng et al. [59] have developed optimal test plans for step-stress accelerated degradation data under a gamma degradation process, and have discussed the optimal determination of sample size, measurement frequency, and termination time at each stress level. For this objective, they minimized the asymptotic variance of the MLE of the mean lifetime of the product within a pre-fixed budget. Subsequently, Zhang et al. [70] have considered sample size, stress level, and test time at each stress level as decision variables in the planning of the test while minimizing the mean square error of the estimate of reliability at a specific time. Earlier, Meeter and Meeker [39] discussed optimal ALT plans based on Type-I censored data under location-scale distribution for log-lifetime with a non-constant scale parameter. Recently, Seo et al. [55] have designed accelerated life-test sampling plans under Weibull distribution with non-constant shape and scale parameters. In addition to Type-I censoring, they also studied optimal test plans for Type-II censoring scheme, wherein stress levels are raised by a pre-fixed number of failures instead of by a pre-fixed time, for the specified model. Pascual [50] has developed ALT plans for competing risks data with Type-I censoring under Weibull distribution with known shape parameter, while Ismail and Aly [24] have discussed optimal test plans for failure-step-stress partial ALT under Weibull distribution. Herein, unlike in the typical step-stress ALTs, the partial ALT combines both ordinary

and ALTs. Yang [66] has also discussed design of ALTs for Type-I censoring under Weibull distribution by minimizing the asymptotic variance of the estimate of the warranty cost of products.

In this Chapter, we present a methodology for planning accelerated life-tests in the context of one-shot device testing by assuming a Weibull distribution with non-constant scale and shape parameters for the lifetime distribution. We consider inspection frequency, number of inspections at each stress level, and allocation of the products as decision variables for minimizing the asymptotic variance of the estimate of the reliability of the products at a specific mission time under normal operating conditions within a pre-fixed budget and a termination time.

5.2 Model Description

Suppose that, at testing condition S_i , N_i items are subjected to J types of stress factors with higher-than-usual stress levels and inspected at K_i equally-spaced time points. Specifically, N_{ik} items are drawn and inspected at a specific time IT_{ik} , with $\sum_{k=1}^{K_i} N_{ik} = N_i$. Then, n_{ik} failure items are collected from the inspection. Let us now assume that the lifetime of the product has a Weibull distribution with scale parameter $\alpha_i > 0$ and shape parameter $\eta_i > 0$ at testing condition S_i , wherein the scale and shape parameters are both related to the stress levels $(x_{i1}, x_{i2}, \ldots, x_{iJ})$ through log-linear links of the following forms:

$$\alpha_i = \exp\left(\sum_{j=0}^J a_j x_{ij}\right) \tag{5.1}$$

and

$$\eta_i = \exp\left(\sum_{j=0}^J b_j x_{ij}\right),\tag{5.2}$$

with $x_{i0} \equiv 1$. The corresponding pdf and cdf are

$$f(t;\alpha_i,\eta_i) = \frac{\eta_i t^{\eta_i - 1}}{\alpha_i^{\eta_i}} \exp\left(-\left(\frac{t}{\alpha_i}\right)^{\eta_i}\right), \qquad t > 0, \tag{5.3}$$

and

$$F(t;\alpha_i,\eta_i) = 1 - \exp\left(-\left(\frac{t}{\alpha_i}\right)^{\eta_i}\right), \qquad t > 0, \tag{5.4}$$

respectively. Once again, for the estimation of the parameters of the Weibull distribution, the extreme value distribution for log-lifetimes is used since this belongs to location-scale family of distributions. The location and scale parameters are given by

$$\mu_{i} = \log(\alpha_{i}) = \sum_{j=0}^{J} a_{j} x_{ij}, \qquad (5.5)$$

and

$$\sigma_i = \eta_i^{-1} = \exp\left(-\sum_{j=0}^J b_j x_{ij}\right),\tag{5.6}$$

respectively.

5.3 Asymptotic Variance of the MLE of Reliability

To determine the optimal test plan by minimizing the asymptotic variance of the MLE of reliability at a specific mission time under normal operating conditions, we need the Fisher information matrix for the model parameters, $\boldsymbol{\theta} =$ $(\boldsymbol{a}, \boldsymbol{b}) = \{a_0, \ldots, a_J, b_0, \ldots, b_J\}$, as well as the first derivatives of the reliability estimate with respect to these model parameters. Earlier in Chapter 4, the loglikelihood function of the log-lifetime data from one-shot device testing is presented in Eq. (4.32) as

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{I} \sum_{k=1}^{K_i} \left[n_{ik} \log(1 - e^{-e^{\nu_{ik}}}) - (N_{ik} - n_{ik})e^{\nu_{ik}} \right] + \text{constant}, \quad (5.7)$$

where $\nu_{ik} = \frac{\log(IT_{ik}) - \mu_i}{\sigma_i}$. The Fisher information matrix under a test plan, ξ , consisting of inspection frequency, number of inspections at each condition, and allocation of the devices, is given by

$$\mathbf{I}(\boldsymbol{\theta};\xi) = -E\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}\right] = -E\left[\begin{array}{c}\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{a} \partial \boldsymbol{a}'} & \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{a} \partial \boldsymbol{b}'}\\ \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{b} \partial \boldsymbol{a}'} & \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{b} \partial \boldsymbol{b}'}\end{array}\right].$$
(5.8)

We have discussed the Fisher information matrix under Weibull distribution with scale and shape parameters varying with stress factors in the preceding section. This readily yields the asymptotic variance-covariance matrix of the MLEs of the model parameters as the inverse of the above Fisher information matrix. Furthermore, given the MLEs of the model parameters, $\hat{\theta} = (\hat{a}, \hat{b})$, the MLE of the reliability of the product at a specific mission time, t_0 , under normal operating conditions, $\boldsymbol{x}_0 = \{x_{01}, x_{02}, \dots, x_{0J}\}$, can be immediately computed as

$$\hat{R}(t_0; \boldsymbol{x}_0) = \exp\left(-\left(\frac{t_0}{\hat{\alpha}_0}\right)^{\hat{\eta}_0}\right), \qquad (5.9)$$

with $\hat{\alpha}_0 = \exp\left(\sum_{j=0}^J \hat{a}_j x_{0j}\right)$ and $\hat{\eta}_0 = \exp\left(\sum_{j=0}^J \hat{b}_j x_{0j}\right)$. We then obtain the corresponding asymptotic variance of the MLE $\hat{R}(t_0; \boldsymbol{x}_0)$ in Eq. (5.9) by using delta method, which has also explained in the preceding section.

5.4 Optimal Test Plan

In this section, to design an efficient accelerated life-test plan for one-shot device testing, we consider the optimization problem of determining the inspection frequency, number of inspections at each condition, and allocation of the devices by minimizing the asymptotic variance of the MLE of the reliability at a specific mission time under normal operating conditions subject to a specified budget and a termination time.

Suppose the budget for conducting an accelerated life-test for one-shot device testing, the operation cost at testing condition S_i , the cost of devices (including the purchase of and testing cost), and the termination time are C_{budget} , $C_{oper,i}$, C_{item} and T_{ter} , respectively. Given a test plan, ξ , that involves the inspection frequency, f, the number of inspections at testing condition S_i , $K_i \ge 2$, and the allocation of devices, N_{ik} , for i = 1, 2, ..., I, the total cost of conducting the experiment is seen to be

$$TC(\xi) = C_{item} \sum_{i=1}^{I} \sum_{k=1}^{K_i} N_{ik} + \sum_{i=1}^{I} C_{oper,i} K_i f.$$
(5.10)

5.4.1 Subject to Specified Budget and Termination Time

The objective of the optimal test plan is to minimize the asymptotic variance of the MLE of the reliability at mission time t_0 under normal operation conditions \boldsymbol{x}_0 , $AVar(\hat{R}(t_0; \boldsymbol{x}_0)|\xi)$, subject to the constraints

$$TC(\xi) \le C_{budget} \tag{5.11}$$

and

$$K_i f \le T_{ter}.\tag{5.12}$$

Due to the complex form of the objective function, we propose the following algorithm for determining the optimal test plan. Let $N_{min} = 20$ (say) be the minimum number of devices allocated at each condition at each inspection time. Then, we follow the following steps:

Step 1: Set
$$f = 1$$
;

Step 2: Find
$$K_i^* = \min\left(\left\lfloor \frac{C_{budget} - (C_{item}N_{min}I + \sum_{i=1}^{I}C_{oper,i}f)}{C_{item}N_{min} + C_{oper,i}}\right\rfloor + 1, \left\lfloor \frac{T_{ter}}{f} \right\rfloor\right),$$

for $i = 1, 2, \dots, I;$

Step 3: For i = 1, 2, ..., I, and $K_i = 2, 3, ..., K_i^*$, let us denote an initial plan with the inspection frequency, the numbers of inspections, and the allocation of the devices with minimum number by $\xi(f, \mathbf{K}, \mathbf{N}_{min})$, where $\mathbf{K} = \{K_i, i =$ $1, 2, ..., I\}$ and $\mathbf{N}_{min} = \{N_{ik} = N_{min}, i = 1, 2, ..., I, k = 1, 2, ..., K_i\}$. For the initial plan $\xi(f, \mathbf{K}, \mathbf{N}_{min})$, if

$$C_{budget} \ge TC(\xi(f, \mathbf{K}, \mathbf{N}_{min})), \tag{5.13}$$

we proceed to the next step in finding the optimal allocation of the devices; otherwise, we skip the next step and jump to Step 5;

- Step 4: The optimal allocation of the devices is determined by a sequential approach in which, starting with the initial test plan $\xi^* = \xi(f, \mathbf{K}, \mathbf{N}_{min})$, an additional device is considered in the test plan and then choosing the best between the new test plan (based on the current one) and the best test plan. Thus, beginning with the initial test plan with $\sum_{i=1}^{I} K_i N_0$ devices,
 - Step(a) $\sum_{i=1}^{I} K_i$ test plans with 1 additional device based on ξ^* are developed, and denoted by ξ_{ik} for i = 1, 2, ..., I and $k = 1, 2, ..., K_i$;
 - Step(b) compute $AVar(\hat{R}(t_0; \boldsymbol{x}_0) | \xi_{ik})$, for i = 1, 2, ..., I and $k = 1, 2, ..., K_i$;
 - Step(c) a new test plan with optimal allocation of the devices can be obtained as $min_{\xi_{ik}}AVar(\hat{R}(t_0; \boldsymbol{x}_0)|\xi_{ik})$, and denoted by $\xi^* = \xi(f, \mathbf{K}, \mathbf{N})$;
 - Step(d) repeat steps (a) to (c) until

$$\sum_{i=1}^{I} \sum_{k=1}^{K_i} N_{ik} = \left\lfloor \frac{C_{budget} - \sum_{i=1}^{I} C_{oper,i} K_i f}{C_{item}} \right\rfloor,$$
(5.14)

and denote the test with the optimal allocation of the devices based on $\xi(f, \mathbf{K}, \mathbf{N}_{min})$ by $\xi^*(f, \mathbf{K}, \mathbf{N})$;

Step 5: Set f = f + 1;

Step 6: Repeat Steps 2 to 5 until $f = T_{ter}$;

Step 7: The optimal solution of $(f, \mathbf{K}, \mathbf{N})$ is then obtained as

$$min_{\xi^*(f,\mathbf{K},\mathbf{N})}AVar(R(t_0;\boldsymbol{x}_0)|\xi^*(f,\mathbf{K},\mathbf{N})).$$

5.4.2 Subject to Standard Error and Termination Time

On the other hand, when the standard error of the estimate of MLE of the reliability $se(\hat{R}(t_0; \boldsymbol{x}_0))$ is given, one may be interested in minimizing the cost of conducting the experiment. For this reason, we propose here the following algorithm for determining the optimal test plan. Again, let $N_{min} = 20, N_{max} = 100$ (say) be the minimum and maximum numbers of devices allocated at each condition at each inspection time. Then, we follow the following steps:

Step 1: Set f = 1;

Step 2: Find

$$K_i^* = \left\lfloor \frac{T_{ter}}{f} \right\rfloor, \tag{5.15}$$

for i = 1, 2, ..., I;

Step 3: For i = 1, 2, ..., I, and $K_i = 2, 3, ..., K_i^*$, let us denote a maximal plan with the inspection frequency, the numbers of inspections, and the allocation of the devices with maximum number by $\xi(f, \mathbf{K}, \mathbf{N}_{max})$, where $\mathbf{N}_{max} = \{N_{ik} = N_{max}, i = 1, 2, ..., I, k = 1, 2, ..., K_I\}$. For the maximal plan $\xi(f, \mathbf{K}, \mathbf{N}_{max})$, if

$$\sqrt{AVar(\hat{R}(t_0; \boldsymbol{x}_0) | \xi(f, \mathbf{K}, \mathbf{N}_{max}))} \le se(\hat{R}(t_0; \boldsymbol{x}_0)), \qquad (5.16)$$

we proceed to the next step in finding the optimal allocation of the devices; otherwise, we skip the next step and jump to Step 5;

- Step 4: The optimal allocation of the devices is determined by a sequential approach in which, starting with the initial test plan $\xi^* = \xi(f, \mathbf{K}, \mathbf{N}_{min})$, with minimum number of devices allocated at each condition at each inspection time. Once again, an additional device is considered in the test plan and then choosing the best between the new test plan (based on the current one) and the best test plan. Thus, beginning with the initial test plan with $\sum_{i=1}^{I} K_i N_0$ devices,
 - Step(a) $\sum_{i=1}^{I} K_i$ test plans with 1 additional device based on ξ^* are developed, and denoted by ξ_{ik} for i = 1, 2, ..., I and $k = 1, 2, ..., K_i$;
 - Step(b) compute $AVar(\hat{R}(t_0; x_0)|\xi_{ik})$, for i = 1, 2, ..., I and $k = 1, 2, ..., K_i$;
 - Step(c) a new test plan with optimal allocation of the devices can be obtained as $min_{\xi_{ik}}AVar(\hat{R}(t_0; \boldsymbol{x}_0)|\xi_{ik})$, and denoted by $\xi^* = \xi(f, \mathbf{K}, \mathbf{N})$;
 - Step(d) repeat steps (a) to (c) until

$$\sqrt{AVar(\hat{R}(t_0; \boldsymbol{x}_0) | \boldsymbol{\xi}^*)} \le se(\hat{R}(t_0; \boldsymbol{x}_0)), \qquad (5.17)$$

and denote the test with the optimal allocation of the devices based on $\xi(f, \mathbf{K}, \mathbf{N})$ by $\xi^*(f, \mathbf{K}, \mathbf{N})$;

Step 5: Set f = f + 1;

Step 6: Repeat Steps 2 to 5 until $f = T_{ter}$;

Step 7: The optimal solution of $(f, \mathbf{K}, \mathbf{N})$ is then obtained as $min_{\mathcal{E}^*(f, \mathbf{K}, \mathbf{N})} A Var(\hat{R}(t_0; \boldsymbol{x}_0) | \boldsymbol{\xi}^*(f, \mathbf{K}, \mathbf{N})).$

5.5 Illustrative Example

In this section, we illustrate the proposed algorithm with an example involving accelerated life-test with 1 stress factor at 3 levels under different budgets and different termination times. Suppose the lifetime of the devices under stress level x_i has a Weibull distribution with scale parameter α_i and shape parameter η_i , where $\alpha_i = \exp(a_0 + a_1 x_i)$ and $\eta_i = \exp(b_0 + b_1 x_i)$. Also, suppose the devices have long lives under standard condition with $(a_0, a_1, b_0, b_1) = (5.7, -0.05, -0.6, 0.03)$. Now, we consider the estimation of the reliability of the devices at mission time $t_0 = 60$ under normal operating stress level of $x_0 = 25$ and the accelerated life-test has to be terminated at times 36 and 60. The elevated stress levels used are 30, 40, and 50, and suppose the costs of operation at these elevated stress levels are \$100, \$150, and \$200 per unit of time, respectively, and the cost of each device (including the purchase and testing) is \$1,100.

5.5.1 Optimal Accelerated Life-Test Plans

The optimal accelerated life-test plans for different budget constraints were determined by using the proposed algorithm, and they are presented in Table 5.1. The corresponding theoretical standard deviation and the root mean square error (based on 2,000 simulated samples) for the estimate of the reliability at time $t_0 = 60$ were also obtained, and these are presented in the last two columns of Table 5.1. The results show that an experiment with sufficiently large time and budget for the experiment, as one would expect, would give better reliability prediction.

Table 5.1: Optimal accelerated life-test plans for one-shot device testing under different budgets and termination times, along with the corresponding standard deviation, *Std*, and root mean square error, *RMSE*, of the MLE of the reliability at mission time $t_0 = 60$.

T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	Std	RMSE
36	\$200,000	(20,34)	(20, 49)	(24, 20)	18	\$199,900	0.0859	0.1002
36	\$300,000	(20, 65)	(20, 89)	(44, 20)	18	\$300,000	0.0634	0.0803
36	\$500,000	(20,126)	(20, 168)	(85,20)	18	\$499,100	0.0462	0.0583
60	\$200,000	(20, 20, 44)	(20, 20)	(20, 20)	19	\$199,400	0.0629	0.0643
60	\$300,000	(20, 20, 20, 41, 78)	(20, 20)	(20, 20)	12	\$299,300	0.0446	0.0606
60	\$500,000	(20, 20, 57, 248)	(20, 20)	(36, 20)	13	\$499,400	0.0319	0.0413

Table 5.2 presents the effect on the optimal accelerated life-test plans when the cost of operation at the highest stress level increases to \$500. We observe that the results in Tables 5.1 and 5.2 are quite close, except for the total number of test devices. Since the cost of operation becomes higher in the latter case, the total number of available devices to be tested becomes slightly less, but it has little impact on the asymptotic variance of the MLE of the reliability. Consequently, both standard derivation and RMSE of the MLE of the reliability increase slightly.

We determine here the minimum costs of conducting an accelerated life-test

Table 5.2: Optimal accelerated life-test plans for one-shot device testing with high cost at higher stress level under different budgets and termination times, along with the corresponding standard deviation, Std, and root mean square error, RMSE, of the MLE of the reliability at mission time $t_0 = 60$.

T_{ter}	C_{budget}	N_{1k}	N_{2k}	N_{3k}	f	TC	Std	RMSE
36	\$200,000	(20, 30)	(20, 45)	(22, 20)	18	\$199,700	0.0902	0.1069
36	\$300,000	(20,61)	(20, 85)	(42, 20)	18	\$299,800	0.0651	0.0816
36	\$500,000	(20, 123)	(20, 164)	(83,20)	18	\$500,000	0.0467	0.0596
60	\$200,000	(20,20,34)	(20, 20)	(20, 20)	19	\$199,800	0.0672	0.0681
60	\$300,000	(20, 20, 20, 35, 78)	(20, 20)	(20, 20)	12	\$299,900	0.0454	0.0608
60	\$500,000	(20, 20, 55, 244)	(20,20)	(35,20)	13	\$499,500	0.0322	0.0416

for different standard errors of the estimate of the reliability at time $t_0 = 60$ by using the proposed algorithm, and the constraints of the termination times and the standard errors for the estimate of the reliability, the accelerated life-test plans, the minimum costs, and the root mean square error (based on 2,000 simulated samples) for the estimate of the reliability are presented in Table 5.3. The results show that the termination time of the experiment is a significant factor in the total cost of conducting the experiment. Since the operation cost is relatively lower than the cost of devices, as one would expect, an experiment with a reasonable time would reduce the total number of test devices as well as the total cost of conducting the experiment.

Table 5.3: Optimal accelerated life-test plans for one-shot device testing under different standard errors for the MLE of the reliability at a mission time $t_0 = 60$ and termination times, along with the corresponding minimum total cost of conducting the experiment, and root mean square error, RMSE, of the MLE of the reliability at the mission time.

T_{ter}	Std	N_{1k}	N_{2k}	N_{3k}	f	TC	RMSE
36	0.10	(20,22)	(20, 37)	(20, 20)	18	\$169,100	0.1152
36	0.08	(20, 39)	(20, 57)	(28, 20)	18	\$218,600	0.0953
36	0.05	(20,107)	(20, 144)	(72, 20)	18	\$437,500	0.0625
60	0.10	(20,20)	(20, 20)	(20, 20)	21	\$150,900	0.1015
60	0.08	(20, 29)	(20, 20)	(20, 20)	25	\$164,400	0.0793
60	0.05	(20, 20, 20, 82)	(20, 20)	(20, 20)	14	\$259,600	0.0530

5.5.2 Sensitivity Analysis over Parameter Misspecification

The effect of misspecification of the model parameters on the optimal test plan need to be studied in order to evaluate the robustness feature of the optimal test plan to misspecification of the model parameters. Since the estimated model parameters $\boldsymbol{\theta} = (\hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1)$ are likely to depart from the true model parameters (a_0, a_1, b_0, b_1) , we assume here that the estimates of the true parameters to be with error such that $\hat{\boldsymbol{\theta}} = (\hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1) = ((1 + \epsilon_1)a_0, (1 + \epsilon_2)a_1, (1 + \epsilon_3)b_0, (1 + \epsilon_4)b_1),$ where $\epsilon_i = \{-0.1, 0.0, 0.1\}$, thus allowing for under-estimation as well as overestimation. Subject to the total cost budget of \$200,000 and termination time of 36, and with the same setup as in the illustrative example in the preceding section, the optimal test plans under various combinations of errors were determined, and these are presented in Table 5.4. Moreover, the relative bias on the estimate of the reliability at the mission time with $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$, given by

$$RB(R(t_0)) = \frac{\hat{R}(t_0; \boldsymbol{x}_0, \boldsymbol{\theta}) - \hat{R}(t_0; \boldsymbol{x}_0, \hat{\boldsymbol{\theta}})}{\hat{R}(t_0; \boldsymbol{x}_0, \boldsymbol{\theta})},$$
(5.18)

were computed and these are presented in the last column of Table 5.4. From the results in Table 5.4, we observe that the determined optimal test plans under misspecification of parameters are quite close to the optimal test plan under correct parameter specification, thus revealing the natural robustness of the developed optimal test plan.

5.6 Concluding Remarks

We have developed here an algorithm for the determination of an optimal test plan for accelerated life-test on one-shot devices by assuming the Weibull lifetime distribution with non-constant scale and shape parameters. This algorithm can also be applied for binary data in survival analysis such as those arising from tumorigenicity experiments wherein the data collected are once again left- and right-censored at each observation time. We have evaluated the performance of this algorithm by determining the optimal test plans subject to different choices

Table 5.4: Sensitivity analysis of optimal accelerated life-tests under various combinations of parameters $((1 + \epsilon_1)a_0, (1 + \epsilon_2)a_1, (1 + \epsilon_3)b_0, (1 + \epsilon_4)b_1)$, with ϵ_i 's being departures from the true values.

ϵ_1	ϵ_2	ϵ_3	ϵ_4	N_{1k}	N_{2k}	N_{3k}	f	$RB(R(t_0))$
0%	0%	0%	0%	(20,34)	(20, 49)	(24, 20)	18	
-10%	-10%	-10%	-10%	(20,37)	(20, 50)	(20, 20)	18	-0.3599
-10%	-10%	-10%	0%	(20,39)	(20, 47)	(21, 20)	18	-0.3656
-10%	-10%	-10%	+10%	(20,20,20)	(20, 20, 29)	(20, 20)	12	-0.3716
-10%	-10%	0%	-10%	(20,36)	(20, 51)	(20, 20)	18	-0.3557
-10%	-10%	0%	0%	(20,37)	(20, 50)	(20, 20)	18	-0.3610
-10%	-10%	0%	+10%	(20,42)	(20, 43)	(21, 21)	18	-0.3667
-10%	-10%	+10%	-10%	(20,36)	(20, 51)	(20, 20)	18	-0.3517
-10%	-10%	+10%	0%	(20, 35)	(20, 52)	(20, 20)	18	-0.3567
-10%	-10%	+10%	+10%	(20,38)	(20, 48)	(21, 20)	18	-0.3621
-10%	0%	-10%	-10%	(20,48)	(20, 37)	(22, 20)	18	-0.4602
-10%	0%	-10%	0%	(20,47)	(20, 36)	(24, 20)	18	-0.4733
-10%	0%	-10%	+10%	(20, 45)	(20, 38)	(24, 20)	18	-0.4874
-10%	0%	0%	-10%	(20,46)	(20, 39)	(22, 20)	18	-0.4503
-10%	0%	0%	0%	(20,46)	(20, 37)	(24, 20)	18	-0.4627
-10%	0%	0%	+10%	(20, 45)	(20, 38)	(24, 20)	18	-0.4761

ϵ_1	ϵ_2	ϵ_3	ϵ_4	N_{1k}	N_{2k}	N_{3k}	f	$RB(R(t_0))$
0%	0%	0%	0%	(20,34)	(20, 49)	(24, 20)	18	
-10%	0%	+10%	-10%	(20,45)	(20, 41)	(21, 20)	18	-0.4410
-10%	0%	+10%	0%	(20,46)	(20, 38)	(23, 20)	18	-0.4527
-10%	0%	+10%	+10%	(20,45)	(20, 38)	(24, 20)	18	-0.4653
-10%	+10%	-10%	-10%	(20,59)	(20, 28)	(20, 20)	18	-0.5565
-10%	+10%	-10%	0%	(20,63)	(20, 24)	(20, 20)	18	-0.5761
-10%	+10%	-10%	+10%	(20,66)	(20, 21)	(20, 20)	18	-0.5970
-10%	+10%	0%	-10%	(20,58)	(20, 29)	(20, 20)	18	-0.5416
-10%	+10%	0%	0%	(20,60)	(20, 27)	(20, 20)	18	-0.5603
-10%	+10%	0%	+10%	(20,66)	(20, 21)	(20, 20)	18	-0.5802
-10%	+10%	+10%	-10%	(20,57)	(20, 30)	(20, 20)	18	-0.5275
-10%	+10%	+10%	0%	(20,58)	(20, 29)	(20, 20)	18	-0.5453
-10%	+10%	+10%	+10%	(20,62)	(20, 25)	(20, 20)	18	-0.5642
0%	-10%	-10%	-10%	(20,32)	(20, 52)	(23, 20)	18	0.0883
0%	-10%	-10%	0%	(20,30)	(20, 53)	(24, 20)	18	0.1150
0%	-10%	-10%	+10%	(20,28)	(20, 53)	(26, 20)	18	0.1431
0%	-10%	0%	-10%	(20,33)	(20, 52)	(22, 20)	18	0.0681
0%	-10%	0%	0%	(20,30)	(20, 53)	(24, 20)	18	0.0935
0%	-10%	0%	+10%	(20,28)	(20, 54)	(25, 20)	18	0.1205

ϵ_1	ϵ_2	ϵ_3	ϵ_4	N_{1k}	N_{2k}	N_{3k}	f	$RB(R(t_0))$
0%	0%	0%	0%	(20, 34)	(20, 49)	(24, 20)	18	
0%	-10%	+10%	-10%	(20, 33)	(20, 52)	(22, 20)	18	0.0488
0%	-10%	+10%	0%	(20, 31)	(20, 53)	(23, 20)	18	0.0731
0%	-10%	+10%	+10%	(20, 28)	(20, 54)	(25, 20)	18	0.0988
0%	0%	-10%	-10%	(20, 35)	(20, 48)	(24, 20)	18	-0.0041
0%	0%	-10%	0%	(20, 34)	(20, 49)	(24, 20)	18	0.0168
0%	0%	-10%	+10%	(20, 33)	(20, 49)	(25, 20)	18	0.0391
0%	0%	0%	-10%	(20, 36)	(20, 48)	(23, 20)	18	-0.0198
0%	0%	0%	+10%	(20, 33)	(20, 50)	(24, 20)	18	0.0212
0%	0%	+10%	-10%	(20, 36)	(20, 48)	(23, 20)	18	-0.0348
0%	0%	+10%	0%	(20, 34)	(20, 49)	(24, 20)	18	-0.0160
0%	0%	+10%	+10%	(20, 33)	(20, 50)	(24, 20)	18	0.0041
0%	+10%	-10%	-10%	(20,41)	(20, 43)	(23, 20)	18	-0.1010
0%	+10%	-10%	0%	(20, 39)	(20, 44)	(24, 20)	18	-0.0868
0%	+10%	-10%	+10%	(20, 39)	(20, 44)	(24, 20)	18	-0.0716
0%	+10%	0%	-10%	(20,41)	(20, 43)	(23,20)	18	-0.1116
0%	+10%	0%	0%	(20,39)	(20, 44)	(24, 20)	18	-0.0982
0%	+10%	0%	+10%	(20,38)	(20, 45)	(24, 20)	18	-0.0839

ϵ_1	ϵ_2	ϵ_3	ϵ_4	N_{1k}	N_{2k}	N_{3k}	f	$RB(R(t_0))$
0%	0%	0%	0%	(20,34)	(20, 49)	(24, 20)	18	
0%	+10%	+10%	-10%	(20,41)	(20, 43)	(23, 20)	18	-0.1217
0%	+10%	+10%	0%	(20,40)	(20, 44)	(23, 20)	18	-0.1090
0%	+10%	+10%	+10%	(20,38)	(20, 45)	(24, 20)	18	-0.0954
+10%	-10%	-10%	-10%	(20,39)	(20, 45)	(23, 20)	18	0.4349
+10%	-10%	-10%	0%	(20,38)	(20, 44)	(25, 20)	18	0.4739
+10%	-10%	-10%	+10%	(20,37)	(20, 43)	(27, 20)	18	0.5132
+10%	-10%	0%	-10%	(20,40)	(20, 45)	(22, 20)	18	0.4040
+10%	-10%	0%	0%	(20,38)	(20, 45)	(24, 20)	18	0.4427
+10%	-10%	0%	+10%	(20,37)	(20, 44)	(26, 20)	18	0.4818
+10%	-10%	+10%	-10%	(20,40)	(20, 45)	(22, 20)	18	0.3734
+10%	-10%	+10%	0%	(20,39)	(20, 45)	(23, 20)	18	0.4116
+10%	-10%	+10%	+10%	(20,38)	(20, 44)	(25, 20)	18	0.4505
+10%	0%	-10%	-10%	(20,39)	(20, 43)	(25, 20)	18	0.3701
+10%	0%	-10%	0%	(20,38)	(20, 43)	(26, 20)	18	0.4082
+10%	0%	-10%	+10%	(20,37)	(20, 42)	(28, 20)	18	0.4470
+10%	0%	0%	-10%	(20,40)	(20, 43)	(24, 20)	18	0.3402
+10%	0%	0%	0%	(20,38)	(20, 43)	(26, 20)	18	0.3777
+10%	0%	0%	+10%	(20,37)	(20, 43)	(27, 20)	18	0.4159

ϵ_1	ϵ_2	ϵ_3	ϵ_4	N_{1k}	N_{2k}	N_{3k}	f	$RB(R(t_0))$
0%	0%	0%	0%	(20, 34)	(20, 49)	(24, 20)	18	
+10%	0%	+10%	-10%	(20, 40)	(20, 43)	(24, 20)	18	0.3110
+10%	0%	+10%	0%	(20, 38)	(20, 44)	(25, 20)	18	0.3476
+10%	0%	+10%	+10%	(20, 38)	(20, 43)	(26, 20)	18	0.3852
+10%	+10%	-10%	-10%	(20, 42)	(20, 40)	(25, 20)	18	0.2990
+10%	+10%	-10%	0%	(20, 41)	(20, 40)	(26, 20)	18	0.3352
+10%	+10%	-10%	+10%	(20, 41)	(20, 39)	(27, 20)	18	0.3726
+10%	+10%	0%	-10%	(20, 42)	(20, 40)	(25, 20)	18	0.2708
+10%	+10%	0%	0%	(20, 41)	(20, 40)	(26, 20)	18	0.3061
+10%	+10%	0%	+10%	(20, 41)	(20, 40)	(26, 20)	18	0.3426
+10%	+10%	+10%	-10%	(20, 42)	(20, 40)	(25, 20)	18	0.2434
+10%	+10%	+10%	0%	(20,41)	(20,41)	(25, 20)	18	0.2777
+10%	+10%	+10%	+10%	(20, 41)	(20, 40)	(26, 20)	18	0.3133

of budget and termination time. We have also carried out a sensitivity analysis to display the natural robustness feature of the developed optimal test plan to misspecification of the model parameters.

The optimal plan test developed here can also be modified to that of a plan based on warranty cost considerations, as done by Yang [67] for accelerated lifetests based on Type-I censoring under the Weibull distribution.

Chapter 6

Concluding Comments

In this thesis, we have considered the one-shot device testing data under accelerated life-test, which is an extreme case of interval censored data. We have developed the EM algorithm for the determination of the MLEs of the model parameters as well as for estimating the reliability at a mission time and the mean lifetime of products under normal operating conditions. In addition to point estimation, the interval estimation of some lifetime parameters of interest is also discussed.

6.1 Summary

The EM algorithm approach is compared to the Bayesian approach under the exponential distribution with single stress model in Chapter 2. The simulation study reveals that, in small sample sizes, the EM algorithm performs quite well for the estimation of the reliability at a mission time in the cases of moderate and low reliability in terms of bias and mean square error. In the case of high reliability, the Bayesian approach with reliable prior information performs better. In general, the Bayesian approach yields more accurate estimate of mean lifetime of products than the EM algorithm does. When the sample size gets larger, both methods result in estimates that are quite comparable.

In Chapter 3, the exponential distribution with multiple stress model is considered and the jackknife technique is employed to reduce the bias of the estimates of parameters of interest. Also, the use of the observed Fisher information matrix, the jackknife technique, the bootstrap method, and the transformation method for the construction of confidence intervals are compared through a Monte Carlo simulation study in terms of coverage probability and average width. It is observed that the distributions of relevant pivotal quantities for the reliability at a mission time and the mean lifetime are quite skewed in small sample sizes, and so the confidence intervals by the use of the observed Fisher information matrix and the jackknife technique, which require normality probabilities for the distribution, do not posses satisfactory coverage probability. Moreover, the bootstrap method for the construction of confidence intervals demands heavy computational time. Hence, the transformation methods for the reliability at a mission time based on logit-transformation and for the mean lifetime based on the log-transformation are observed to perform satisfactorily, and so are the ones that are recommended.

Chapter 4 discusses the Weibull distribution with multiple stress model and the

determination of the MLEs of the model parameters by using the EM algorithm, and also construction of various confidence intervals. Since the lifetime model involves scale as well as shape parameters that are both allowed to vary with the stress factors, a large sample size is required for fitting this model to data; otherwise, the iterative estimation method may not converge. Then, a one-shot device testing data arising from survival study involving a tumorigenicity experiment is analyzed. The analysis shows that the covariates of the gender, the strain of offspring, and the concentration of the chemical of benzidine dihydrochloride in the drinking water have significant effects on both scale and shape parameters of the Weibull distribution. In other words, the shape of the Weibull lifetime distribution varies with all these three covariates, and so a conventional proportional hazards analysis based on Weibull distribution will be erroneous is this case.

Since the asymptotic variance of the estimate of the reliability at a mission time under the Weibull distribution has been derived, in Chapter 5, we have used it to develop an algorithm to obtain an optimal accelerated life-test plan by minimizing the asymptotic variance subject to constraints on the budget and the termination time of the life-test. In addition, with a given standard error of the estimates of the reliability at a mission time, we can determine the budget and design the test plan for collecting the data. We have carried out a sensitivity analysis to check the robustness of the developed optimal test plan. This analysis displays that the proposed optimal test plan is stable even if there are moderate departures from specified model parameters.

6.2 Further Generalizations and Extensions

For further work, we may consider the gamma and lognormal distributions as models for lifetimes of devices, and then develop the corresponding likelihood inferential results and optimal test plans based on the asymptotic variance of the estimate of the reliability at a mission time. Note that the gamma distribution would also generalize the results developed in Chapters 2 and 3 based on the exponential distribution. Furthermore, we may study of the effect of model misspecification on the optimal design of test plan; for example, when the data is treated wrongly to fit a gamma model when the true lifetime of the devices is a Weibull distribution.

We may also develop an optimal test plan based on warranty cost considerations under different models such as inverse power law model and Arrhenius accelerated model that relate the parameters in the lifetime model to voltage and temperature as stress factors, respectively.

To reduce the cost of the experiment by decreasing the number of test devices, we may consider a step-stress accelerated test instead of constant-stress accelerated test. Under such an ALT, it will be of great interest to develop the EM algorithm for the determination of the MLEs of the model parameters and also to develop methods for interval estimation.

Next, in the context of an accelerated life-test, it is reasonable to restrict the model parameters $a_j > 0$. But, the maximization in the M-step becomes more complicated in this case since we need to solve a restricted maximization problem.

It will be of interest to find the restricted MLEs in this case and then develop the corresponding order-restricted inferential results.

A distance-based test statistic and the corresponding approximate *p*-value by bootstrap method have been discussed briefly in Chapter 3 to examine the fit of the exponential model to a data. In this regard, finding a suitable test statistic with good power properties for testing the suitability of the model for the accelerated life-test on one-shot devices remains as an open problem. This will also be of practical interest as model validation is a key aspect of any statistical analysis.
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