BRAUER-KURODA RELATIONS FOR HIGHER CLASS NUMBERS

BRAUER-KURODA RELATIONS FOR HIGHER CLASS NUMBERS

By

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Abstract

Arising from permutation representations of finite groups, Brauer-Kuroda relations are relations between Dedekind zeta functions of certain intermediate fields of a Galois extension of number fields. Let E be a totally real number field and let $n \geq 2$ be an even integer. Taking $s = 1 - n$ in the Brauer-Kuroda relations then gives a correspondence between orders of certain motivic and Galois cohomology groups. Following the works of Voevodsky and Wiles (cf. [33], [36]), we show that these relations give a direct relation on the motivic cohomology groups, allowing one to easily compute the higher class numbers, the orders of these motivic cohomology groups, of fields of high degree over $\mathbb{Q}$ from the corresponding values of its subfields. This simplifies the process by restricting the computations to those of fields of much smaller degree, which we are able to compute through Sage ([30]). We illustrate this with several extensive examples.

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0.1 Introduction

Dirichlet was the the first to establish a relation between class numbers of a number field and its subfields in 1842, showing that for a square-free positive integer $m > 1$, the class number of $\mathbb{Q}(\sqrt{m}, \sqrt{-m})$ is either the product of the class numbers of $\mathbb{Q}(\sqrt{m})$ and $\mathbb{Q}(\sqrt{-m})$ or half of that. In 1950, Brauer and Kuroda independently initiated a systematic study of the relations between class numbers in number fields arising from isomorphisms of permutation representations of finite groups. In particular, they showed how, in a Galois extension of number fields, relations between permutation characters of subgroups provide relations between invariants such as the discriminant, class number and regulator, of the corresponding intermediate fields.

Let E be an algebraic number field with r_1 real embeddings and r_2 pairs of conjugate complex embeddings. Let o_E denote the ring of integers in E and $\zeta_E(s)$ its Dedekind zeta function, defined as

$$\zeta_E(s) = \sum_{0 \neq I \subseteq o_E} \frac{1}{N(I)^s},$$

where the sum extends over all non-zero ideals $I \subseteq o_E$ and $N(I) = |o_E/I|$. As the generalization of the Riemann zeta function, the Dedekind zeta function admits a functional equation relating the values $\zeta_E(s)$ and $\zeta_E(1-s)$. For an integer $n \geq 1$, this functional equation yields the order of vanishing of $\zeta_E(s)$ at $s = 1-n$, denoted d_n as follows

$$d_n = \begin{cases} r_1 + r_2 - 1, & \text{if } n = 1 \\ r_1 + r_2, & \text{if } n \geq 3 \text{ is odd} \\ r_2, & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

We denote by $\zeta_E^*(s)$ the first non-vanishing coefficient in a Taylor expansion around s and call this the *special value of the zeta function* at s. Taking $s = 0$ then yields

Theorem.

$$\zeta_E^*(0) = -\frac{h_E}{w_E} \cdot R_E,$$

where w_E denotes the number of roots of unity in E, h_E denotes the class number of E, and R_E is the Dirichlet regulator.

This formula, due to Dirichlet, is known as the *Analytic Class Number Formula.* Note that the value $\zeta_E(0)$ is precisely non-zero when $E = \mathbb{Q}$ or E is an imaginary quadratic field, so that in this case, the Analytic Class Number Formula states

$$\zeta_E(0) = \zeta_E^*(0) = -\frac{h_E}{w_E}.$$

Similarly, for real quadratic fields E, we have $w(E) = 2$ and $R_E = \log \varepsilon(E)$, where $\varepsilon(E) > 1$ is the fundamental unit of the field E. Thus the Analytic Class Number Formula takes the form

$$\zeta_E^*(0) = -\frac{1}{2} \log \varepsilon(E) \cdot h_E.$$

Results such as these, concerning the case $s = 0$, are classical in the sense that, generally, there are known effective algorithms for computing the values of the class number h_E and of the regulator R_E of a number field E. Our knowledge in the case where s is a negative odd integer is less complete however, as we shall see.

More precisely, consider now the special values at $s = 1 - n$ for $n \geq 2$. Around 1971, Lichtenbaum gave a conjectural generalization of Dirichlet's Class Number Formula, suggesting that the correct formula should read:

Conjecture. *(Lichtenbaum) For any number field E and all $n \geq 2$,*

$$\zeta_E^*(1-n) = \pm \frac{|K_{2n-2}(o_E)|}{|K_{2n-1}(o_E)_{tors}|} \cdot R_n^B(E),$$

up to powers of 2, where $R_n^B(E)$ denotes the Borel regulator.

This conjecture has been proven to be true in the case that E is an abelian number field ([17]). Namely,

Theorem. *Let E be an abelian number field. For all $n \geq 2$,*

$$\zeta_E^*(1-n) = \pm \frac{h_n(E)}{w_n(E)} \cdot R_n(E),$$

up to powers of 2, where $w_n(E)$ denotes the order of the Galois cohomology group $H^0(E, \mathbb{Q}/\mathbb{Z}(n))$ and $h_n(E)$ denotes the order of the motivic cohomology group $H^2_{\mathcal{M}}(o_E, \mathbb{Z}(n))$ of the ring of integers o_E.

Here, the formulation uses the Beilinson regulator $R_n(E)$ in place of the Borel regulator, although they are the same up to powers of 2 (cf. [15]).

In order to obtain a formulation including the exact power of 2, we must consider motivic cohomology in place of algebraic K-theory. That is, for any number field E and any integer $n \geq 2$, the motivic formulation of the Lichtenbaum Conjecture including the 2-primary information is

$$\zeta_E^*(1-n) = \pm \frac{|H^2_{\mathcal{M}}(E, \mathbb{Z}(n))|}{|H^1_{\mathcal{M}}(E, \mathbb{Z}(n))_{tors}|} \cdot R_n^{\mathcal{M}}(E),$$

where $R_n^{\mathcal{M}}$ is the motivic regulator.

Returning now to the functional equation, it is clear that the value of $\zeta_E(1-n)$ is non-zero precisely when $n \geq 2$ is even and E is totally real. As a consequence of the Main Conjecture of Iwasawa theory, this value is given by

Theorem. *Let E be a totally real number field with ring of integers o_E, and let $n \geq 2$ be an even integer. Then*

$$\zeta_E(1-n) = (-1)^{\frac{n}{2}} \frac{h_n(E)}{w_n(E)},$$

Note that for $n = 1$, the group $H^0(E, \mathbb{Q}/\mathbb{Z}(1))$ consists of the roots of unity in E, so that $w_1(E) = w_E$ in our previous notation.

The Iwasawa Main Conjecture has been proven by Wiles (cf. [36]) for odd primes p and for $p = 2$ in the case that E is abelian over $\mathbb{Q}$, but remains open in general. As such, throughout this paper, we will assume that the general 2-adic Iwasawa Main Conjecture holds.

Note that the values $\zeta_E(1-n)$ for $n \geq 2$ even are typically difficult to compute, and often exceed the capabilities of computing software such as Sage ([30]). This is particularly true of number fields E of high degree over $\mathbb{Q}$, with the complexity of these computations increasing with the degree of the extension. Fortunately, following the works of R. Brauer and S. Kuroda ([5], [18]), certain non-trivial multiplicative relations exist between the zeta functions of a number field E and the zeta functions of a particular proponent of its subfields. Using these relations, we can reduce the computations of the Dedekind zeta function to that of its subfields, which we are able to calculate using Sage. Having calculated $\zeta_E(1-n)$, we can then even

iterate this process and calculate the Dedekind zeta function values of number fields of even larger degrees via these relations.

These so-called *Brauer-Kuroda relations* are a simple consequence of Artin's formalism of L-functions ([1], [2]). More precisely, letting E/F be an extension of algebraic number fields with Galois group $G = Gal(E/F)$, the following multiplicative relation holds:

Theorem. *(Brauer) Let $\mathbf{1}_G$ denote the principal character of G, and, for every cyclic subgroup H of G, let*

$$a(H) = \frac{1}{[G:H]} \sum_{\substack{H^* - cyclic \\ H \subseteq H^* \subseteq G}} \mu([H^* : H]),$$

where the summation is taken over all cyclic subgroups $H^* \supseteq H$ and μ denotes the Möbius function. Then

$$\mathbf{1}_G = \sum_{\substack{H - cyclic \\ H \subseteq G}} a(H) Ind_H^G(\mathbf{1}_H),$$

and such a relation is called a Brauer-Kuroda relation.

Artin's formalism for Artin L-functions then implies:

Theorem. *(Brauer) For every cyclic subgroup H of G, let*

$$a(H) = \frac{1}{[G:H]} \sum_{\substack{H^* - cyclic \\ H \subseteq H^* \subseteq G}} \mu([H^* : H]).$$

Then, letting E^H denote the fixed field of H over F,

$$\zeta_F(s) = \prod_{\substack{H - cyclic \\ H \subseteq G}} \zeta_{E^H}^{a(H)}(s).$$

These relations arise from isomorphisms of permutation representations of finite groups and thereby depend entirely on the structure of the Galois group G of the field extension E/F, and not on the fields themselves. In the sections to follow, we will illustrate this by several examples, some of which are less well-known.

Following Dirichlet's Analytic Class Number formula, substituting $s = 0$ in the Brauer-Kuroda relation then implies a relation between the class numbers, regulators and number of roots of unity of the corresponding fields. Following this, Brauer was able to show further that

Theorem. *(Brauer) If*

$$\sum_H a(H) Ind_H^G(\mathbf{1}_H) = 0$$

denotes a Brauer-Kuroda relation on G, then

$$\prod_H w_1(E^H)^{a(H)} = \prod_H w_1(E^H)(2)^{a(H)},$$

where $w_1(E)$ denotes the number of roots of unity of the field E and $w_1(E)(p)$ its corresponding p-part. Suppose also (ζ_{2^k}) is the group of 2-power roots of unity in E and $F(\zeta_{2^k})/F$ is cyclic. Then

$$\prod_H w_1(E^H)(2)^{a(H)} = 1.$$

Substituting $s = 1 - n$ into the Brauer-Kuroda relation, where $n \geq 2$ is even, we are able to obtain relations between the higher class numbers h_n and values w_n of the corresponding fields. In this way, Bartel, de Smit [4] were able to show, more generally, that given such a relation,

$$\prod_H w_n(E^H)^{a(H)}$$

is a power of two.

In our main theorem, we improve this finding, showing even more generally that

Proposition. *Given a Brauer-Kuroda relation on G,*

$$\sum_H a(H) Ind_H^G = 0,$$

it follows that

$$\prod_H w_n(E^H)^{a(H)} = 1$$

for all n even. If n is odd,

$$\prod_H w_n(E^H)^{a(H)} = \prod_H w_1(E^H)(2)^{a(H)}.$$

Here, we observe that when n is odd, the values $w_n(E^H)$ cancel in the relation according to Brauer's theorem on $w_1(E^H)$. In particular, if E/F is a Galois extension of totally real number fields, then the relation holds for w_n for all n.

Since the Brauer-Kuroda relations hold for $\zeta_E(1-n)$ and $w_n(E)$ up to powers of 2, it follows from the above results on the Dedekind zeta function at negative odd integers that these relations yield a correspondence between the higher class numbers $h_n(E)$, and in the case that E is not totally real, the regulators. Therefore, we can reduce the computation of $w_n(E)$ to those of its corresponding subfields, $w_n(E^H)$, which are easier to calculate, and, by similarly calculating $\zeta_E(1-n)$, determine $h_n(E)$ in the case that E is totally real. We provide several examples of these computations, namely looking at number fields of high degree with multicyclic Galois group.

Lastly, we consider the n-th motivic wild kernel of F, denoted by $WK^{\mathcal{M}}_{2n-2}(E)$, whose order is given by $h^{\mathcal{M}}_n(E)$. This is a canonical subgroup of $H^2_{\mathcal{M}}(o_E, \mathbb{Z}(n))$ determined by local conditions, and is analogous to the Tate-Shafarevich group of an abelian variety A defined over a number field E. As we shall see, we have

$$h^{\mathcal{M}}_n(E) = w_{n-1}(E) \cdot 2^{-d} \cdot \frac{h_n(F)}{\prod_p \prod_{v|p} w_{n-1}(E_v)(p)},$$

where p runs through all primes, v runs through all primes in E above p, and d denotes the degree of the extension E/F.

Analogous to our above work with $w_n(E)$, we attempt to determine whether the values $h^{\mathcal{M}}_n(E)$ cancel in Brauer-Kuroda relations when E is a totally real number fields and n is even. In this case, our previous work shows that the $h_n(E)$ cancel in Brauer-Kuroda relations, as do the values $w_{n-1}(E)$. The difficulty lies in determining whether this is also true of the local factors E_v, which as we shall see through several extensive examples, is largely dependent on how the primes p split in E. Further, we observe that no general formula for the deviation exists, largely in lieu of the local factors now involved, and the many different cases they present.

Chapter 1 provides a brief review of the theory of zeta functions. In Chapter 2, we introduce the Brauer-Kuroda relations, first recalling some necessary linear

representation theory. In Chapter 3, we present the cohomological background necessary for our main result, proved in Chapter 4. We then use Brauer-Kuroda relations to compute the Dedekind zeta functions of several large number fields via our main result in Chapter 5. Additionally, Chapter 6 introduces the motivic wild kernel and its implications in Brauer-Kuroda relations. Finally, we provide our computations of $h_n^{\mathcal{M}}(E)$ in Brauer-Kuroda relations in Chapter 7.

Chapter 1

Zeta Functions

In the first half of the eighteenth century, Euler introduced and studied the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

as a function of a natural argument s, calculating the sums for all even exponents in terms of Bernoulli numbers. In time, this definition was extended by Riemann into what is modernly known as the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where s is a complex variable.

We begin this Chapter by first introducing some of the important properties of the Riemann zeta function, and, once established, will describe a generalization of this function, called the Dedekind zeta function.

Throughout, we will follow the work presented in [25] and [11].

1.1 The Riemann Zeta Function

The *Riemann zeta function* may be expressed by Euler's identity

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

where the product extends over all prime numbers p. This function is absolutely and uniformly convergent in the domain $Re(s) \geq 1 + \delta$ for every $\delta > 0$, and as such represents an analytic function in the half-plane $Re(s) > 1$.

The Riemann zeta function admits a meromorphic continuation throughout the complex plane with a single pole at $s = 1$ and residue

$$\lim_{s \to 1} (s - 1)\zeta(s) = 1.$$

Furthermore, it satisfies a functional equation which relates the argument s to the argument $1 - s$, namely

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1 - s), \tag{1.1}$$

where $\Gamma(s)$ denotes the Γ-function, which is a non-zero meromorphic function with simple poles at $s = -m$ for $m = 0, 1, \ldots$ and residues $(-1)^m/m!$.

1.2 The Dedekind Zeta Function

The Riemann zeta function is associated with the field of rational numbers $\mathbb{Q}$, but can be generalized to an arbitrary number field E of degree $n = [E : \mathbb{Q}]$. This generalization is called the *Dedekind zeta function* and is defined as the Dirichlet series

$$\zeta_E(s) = \sum_{0 \neq I \subset o_E} \frac{1}{N(I)^s},$$

where o_E denotes the ring of integers of E, and for any nonzero ideal $I \subset o_E$, $N(I) = |o_E/I|$ denotes the ideal norm of E, defined as the number of elements in the finite quotient ring o_E/I.

Like the Riemann zeta function, $\zeta_E(s)$ converges absolutely and uniformly on $Re(s) \geq 1+\delta$ for every $\delta > 0$ and also admits a meromorphic continuation to the complex plane with a single simple pole at $s = 1$. Furthermore, we have the Euler product

$$\zeta_E(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}},$$

where the product extends over all prime ideals $\mathfrak{p}$ of o_E.

Due to Dirichlet, the corresponding residue of $\zeta_E(s)$ at $s = 1$, called the *Analytic Class Number Formula*, is given by

$$\lim_{s \to 1} \zeta_E(s) = \frac{2^{r_1+r_2} \cdot \pi^{r_2} \cdot R_E \cdot h_E}{w_E \cdot |d_E|^{1/2}}, \tag{1.2}$$

where h_E is the class number of E, R_E is the regulator of E, w_E is the number of roots of unity in E, and d_E denotes the discriminant of E. Here, also, we note that r_1 is the number of real embeddings of E, while $2r_2$ denotes the number of complex embeddings of E.

Letting

$$A = 2^{-r_2} \pi^{-\frac{r_1+2r_2}{2}} \sqrt{|d_E|},$$

the Dedekind zeta function analogously satisfies a functional equation relating the argument s to $1-s$, which reads

$$A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_E(s) = A^{1-s} \Gamma\left(\frac{1-s}{2}\right)^{r_1} \Gamma(1-s)^{r_2} \zeta_E(1-s). \tag{1.3}$$

Recall that the Gamma function $\Gamma(z)$ is analytic everywhere except at $z = 0, -1, -2, \ldots$. Hence, if $s = n$ is a natural number ≥ 1, then the product of Gamma factors on the right-hand side of the functional equation yields a pole of total order $r_1 + r_2$ if n is odd, and of total order r_2 if n is even. For all $n \geq 2$, the left-hand side at $s = n$ is a positive real number so that the order of vanishing of $\zeta_E(s)$ at $s = 1-n, n \geq 2$, is equal to r_2 if n is even and equal to $r_1 + r_2$ if n is odd. If $s = n = 1$, note that the left-hand side of the functional equation has a simple pole, and hence the order of vanishing of the zeta function of E at $s = 0$ is equal to $r_1 + r_2 - 1$. Letting d_n denote the order of vanishing of the zeta function at $s = 1-n$,

we can summarize this information as

$$d_n = \begin{cases} r_1 + r_2 - 1 & \text{if } n = 1 \\ r_1 + r_2 & \text{if } n \geq 3 \text{ is odd} \\ r_2 & \text{if } n \geq 2 \text{ is even.} \end{cases} \tag{1.4}$$

Consider the Taylor expansion of the Dedekind zeta function around s and let $\zeta_E^*(s)$ denote the first non-vanishing coefficient in this expansion. An important result of Dirichlet uses the Analytic Class Number Formula to compute the special value $\zeta_E^*(0)$ via the functional equation as follows:

Theorem 1.1. *(Dirichlet) For a number field E,*

$$\zeta_E^*(0) = -\frac{h_E}{w_E} \cdot R_E. \tag{1.5}$$

The Analytic Class Number Formula is a deep result connecting a special value of the Dedekind zeta function of a number field E to the class number of E. In the sections to follow, we will show that this result can be extended to other special values of $\zeta_E(s)$, which in turn can be related to other important arithmetic properties of o_E. Numerically speaking, we will also note that the complexity of the computations involving zeta functions and other such special values increases with the degree of the number fields in question. We will show how, using certain Brauer-Kuroda relations, one can compute these values, which are otherwise even outside the capabilities of numerical software such as Sage.

Chapter 2

Brauer-Kuroda Relations

In the 1950's, Brauer and Kuroda showed how to calculate values of the Dedekind zeta function $\zeta_E(s)$ of a Galois extension of number fields E/F using multiplicative relations involving only the Dedekind zeta functions of certain subfields. These relations arise from relations between permutation characters of subgroups of the Galois group $G = Gal(E/F)$.

The aim of this section is to derive these non-trivial identities, collectively known as Brauer-Kuroda relations, and illustrate how they can be used further to determine other invariants of number fields.

The background for the material presented in this chapter is taken primarily from [28] and [8], and the material presented in Section 2.2 can be found in [5]

2.1 Linear Representation Theory

2.1.1 Definition of Representation

Let G be a finite group, let $\mathbb{C}$ denote the field of complex numbers and let V be a finite-dimensional vector space over $\mathbb{C}$. A *linear representation* of G is any

homomorphism $\varphi : G \to GL(V)$, where $GL(V)$ denotes the group of nonsingular linear transformations from V to itself. In other words, we associate to each element $g \in G$ a nonsingular map $\varphi(g) : V \to V$ in such a way that we have the equality

$$\varphi(gh) = \varphi(g) \circ \varphi(h) \quad \text{for } g, h \in G.$$

We observe that the preceding formula implies the following:

$$\varphi(1) = 1, \quad \varphi(g^{-1}) = \varphi(g)^{-1}.$$

Let $n = \dim V$ where $n \in \mathbb{Z}^+$. A *matrix representation* of G is any homomorphism $G \to GL_n(\mathbb{C})$, where $GL_n(\mathbb{C})$ is the group of invertible $n \times n$ matrices with entries in $\mathbb{C}$. Note that, by fixing a basis for V, we obtain an isomorphism $GL(V) \cong GL_n(\mathbb{C})$, so that a linear representation of G gives rise to a matrix representation of G and vice versa, allowing one to pass freely between the two definitions.

Recall that the set of all formal sums of the form

$$\sum_{g \in G} \alpha_g g, \quad \alpha_g \in \mathbb{C}$$

with component-wise addition and multiplication $(\alpha g)(\beta h) = (\alpha\beta)(gh)$ (where $\alpha\beta$ is multiplied in $\mathbb{C}$ and gh is a product in G) extended to sums via the distributive law is called the *group ring* of G over $\mathbb{C}$, denoted $\mathbb{C}[G]$. It is a vector space over $\mathbb{C}$ with the elements of G as a basis, so that necessarily, it is of dimension $|G|$.

There is a bijection between finitely generated $\mathbb{C}[G]$-modules and representations of G, sending a $\mathbb{C}[G]$-module V to the pair (V, φ), where $\varphi : G \to GL(V)$ is the representation of G defined by $\varphi(g)(v) = g \cdot v$ for all $v \in V$. Under this correspondence, we shall say that the module V *affords* the representation φ of G.

Two representations of G are said to be *equivalent* if the $\mathbb{C}[G]$-modules affording them are isomorphic modules. That is to say, if $\varphi : G \to GL(V)$ and $\psi : G \to GL(W)$ are two representations of G, where V and W are vector spaces over $\mathbb{C}$, then these representations are equivalent if there exists a $\mathbb{C}[G]$-module isomorphism $\tau : V \to W$ such that

$$\tau \circ \varphi(g) = \psi(g) \circ \tau \quad \text{for all } g \in G.$$

In the present case, if V is a $\mathbb{C}[G]$-module affording the representation φ, then a subspace W of V is called *G-invariant* or *G-stable* if $w \in W$ implies $\varphi(g)(w) \in W$ for

all $g \in G$. It follows easily that the $\mathbb{C}[G]$-submodules of V are precisely the G-stable subspaces of V. The restriction $\varphi(g)|_W$ of $\varphi(g)$ to W is an isomorphism of W onto itself, and we have that $\varphi(gh)|_W = \varphi(g)|_W \circ \varphi(h)|_W$. Thus $\varphi|_W : G \to GL(W)$ is a linear representation of G in W, and so W is said to be a *subrepresentation* of V.

As above, assume that $\varphi : G \to GL(V)$ is a representation of G in V and let W be a vector subspace of V, stable under G. A theorem by Maschke then states that there exists a complement $\overline{W}$ of W which is stable under G. That is to say, $V = W \oplus \overline{W}$.

We say that a linear representation $\varphi : G \to GL(V)$ of G is *irreducible* if V is nontrivial and if no vector subspace is stable under G except for 0 and V itself. In other words, V is not the direct sum of two nontrivial representations. Indeed, every representation can written as a direct sum of irreducible representations. Of course, this is equivalent to saying that every finitely generated $\mathbb{C}[G]$-module is completely reducible in that it can be written as a direct sum of submodules.

2.1.2 Character Theory

Throughout this section, G is a finite group and all representations considered are finite-dimensional. Let V be an n-dimensional vector space having basis $\{e_i\}$ and let ρ be a linear map of V onto itself with matrix (ρ_{ij}). We recall that the trace of the matrix (ρ_{ij}) is the scalar $\operatorname{tr}\rho = \sum_i \rho_{ii}$, and does not depend on the choice of basis $\{e_i\}$. Now let φ be a representation of G afforded by the $\mathbb{C}[G]$-module V. For each $g \in G$, define the *character* of φ to be the function

$$\chi : G \to \mathbb{C} \quad \text{defined by} \quad \chi(g) = \operatorname{tr}\varphi(g).$$

It is worth noting that two representations with the same character are equivalent, and vice versa, reducing the study of representations to that of their characters. A character of an irreducible representation is called an *irreducible* character. With this in mind, if $\chi_1, \ldots, \chi_k$ are the distinct irreducible characters of G, and if $W_1, \ldots, W_k$ denote corresponding representations, each representation V is isomorphic to a direct sum

$$V = m_1 W_1 \oplus \cdots \oplus m_k W_k,$$

where the coefficients m_i are integers ≥ 0. The character χ of V is then equal to the sum $m_1\chi_1 + \ldots m_k\chi_k$. Hence, as every representation can be expressed uniquely as

a direct sum of irreducible representations, the character of any representation can similarly be expressed as a $\mathbb{N}$-linear combination of irreducible characters.

2.1.3 Induced Characters

Consider a subgroup H of G and let φ be a representation of H over $\mathbb{C}$. From φ, we can obtain a representation of G called the *induced representation*, whose character, as we will show, is defined in terms of φ. However, we first note that it may not be possible to extend φ to a representation ψ of G in such a way that $\psi|_H = \varphi$.

For such a subgroup H of a finite group G, let V be an $\mathbb{C}[H]$-module affording the representation φ of H. The $\mathbb{C}[G]$-module $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is called the *induced module* of V, and the representation of G it affords is called the *induced representation* of φ. If χ is the character of φ, then the character of the induced representation is called the *induced character*, denoted by $Ind_H^G(\chi)$.

More precisely, the induced character is given by

$$Ind_H^G(\chi)(g) = \sum_{i=1}^{m} \chi(g_i^{-1} g g_i),$$

where $\chi(g_i^{-1} g g_i)$ is defined to be 0 if $g_i^{-1} g g_i \notin H$.

Let V be a 1-dimensional vector space over $\mathbb{C}$. Make V into an $\mathbb{C}[G]$-module by letting $g \cdot v = v$ for all $g \in G$ and $v \in V$. This module affords the representation $\varphi : G \to GL(V)$ defined by $\varphi(g) = i$ for all $g \in G$, where i is the identity linear transformation. The corresponding matrix representation is the homomorphism of G into $GL_1(\mathbb{C})$ which sends every group element to the 1×1 identity matrix. We refer to this representation as the *trivial representation*. The character of the trivial representation is then the function $\mathbf{1}_G(g) = 1$ for all $g \in G$, which we refer to as the *principal character* of G.

Let $\mathbf{1}_H$ be the principal character of the subgroup H. Then $Ind_H^G(\mathbf{1}_H)(g)$ counts 1 for each coset representative g_i such that $g_i^{-1} g g_i \in H$. Since $g_i^{-1} g g_i \in H$ if and only if g fixes the left coset $g_i H$ under left multiplication, $Ind_H^G(\mathbf{1}_H)(g)$ is the number of points fixed by g in the permutation representation of g on the left cosets of H. That is, if $\mathbf{1}_H$ is the principal character of H, then $Ind_H^G(\mathbf{1}_H)$ is the permutation character on the left cosets of H in G. In the language of $\mathbb{C}[G]$-modules, this means simply that

$Ind_H^G(\mathbf{1}_H)$ is the character afforded by a $\mathbb{C}[G/H]$-module V. In the sections to follow, we will see that this representation is of central importance to our computations and main results.

2.2 Relations Between Permutation Representations

Let E/F be a Galois extension of number fields with Galois group $G = Gal(E/F)$, and let φ be a representation of G whose corresponding character is denoted by χ. Following Artin's Induction Theorem, χ is a rational linear combination of characters induced from representations of cyclic subgroups of G. More precisely

$$\chi = \sum_H a(H) Ind_H^G(\mathbf{1}_H),$$

where the sum extends over all cyclic subgroups H of G and $a(H) \in \mathbb{Q}$. In 1950, Brauer extended this result by giving a concise definition of the coefficients $a(H)$ above.

Before stating this result, recall briefly that the Möbius function $\mu(n)$, whose domain is the set of all positive integers n, takes its values in $\{-1, 0, 1\}$ depending on the factorization of n into prime factors. It is defined as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is a square-free positive integer with an even number of prime factors} \\ -1 & \text{if } n \text{ is a square-free positive integer with an odd number of prime factors} \\ 0 & \text{if } n \text{ is not square-free.} \end{cases}$$

With this definition in mind, we have

Theorem 2.1. *(Brauer) Let χ be a character of G, and, for every cyclic subgroup H of G, let*

$$a(H) = \frac{1}{[G:H]} \sum_{\substack{H^* - cyclic \\ H \subseteq H^* \subseteq G}} \mu([H^* : H]) \chi(x^*), \tag{2.1}$$

where the summation is taken over all cyclic subgroups $\langle x^* \rangle = H^*$. Then

$$\chi = \sum_{\substack{H - cyclic \\ H \subseteq G}} a(H) Ind_H^G(\mathbf{1}_H). \tag{2.2}$$

Here, in considering the special case where $\chi = \mathbf{1}_G$, we obtain

$$\mathbf{1}_G = \sum_{\substack{H-\text{cyclic} \\ H \subseteq G}} a(H) Ind_H^G(\mathbf{1}_H) \quad \text{where} \quad a(H) = \frac{1}{[G:H]} \sum_{\substack{H^*-\text{cyclic} \\ H \subseteq H^* \subseteq G}} \mu([H^*:H]). \tag{2.3}$$

Such a relation between permutation representations of G will be called a *Brauer-Kuroda* relation.

Given such a relation, Artin's formalism for the Artin L-functions $L(s, Ind_H^G, E/F)$ ([2], [1]) then implies that we have an equality of zeta functions ζ_{E^H} of the corresponding fixed fields, keeping in mind that an additive decomposition of characters corresponds to a multiplicative decomposition of L-functions. In particular:

Theorem 2.2. *(Brauer) Let E/F be a Galois extension of number fields with Galois group $G = Gal(E/F)$. For every cyclic subgroup H of G, let*

$$a(H) = \frac{1}{[G:H]} \sum_{\substack{H^*-cyclic \\ H \subseteq H^* \subseteq G}} \mu([H^*:H]). \tag{2.4}$$

Then, letting E^H denote the fixed field of H over F,

$$\zeta_F(s) = \prod_{\substack{H-cyclic \\ H \subseteq G}} \zeta_{E^H}^{a(H)}(s) \tag{2.5}$$

Remark 1. Artin's formalism also implies that, given two subgroups H and H' of $G = Gal(E/F)$ which give rise to equivalent permutation representations $Ind_H^G \cong Ind_{H'}^G$, the corresponding fixed fields E^H and $E^{H'}$ have the same zeta function.

2.2.1 Examples

In this section, we illustrate how to calculate Brauer-Kuroda relations for some simple examples. In doing so, we will observe that these relations depend essentially on the structure of the Galois group of the extension E/F, and not on the fields E, F. Here, we take $F = \mathbb{Q}$.

Example 1. Let G be the cyclic group of order n. Then, for every $d|n$, there exists a unique cyclic subgroup H of order d. Iterating this argument, we note that every subgroup H^* containing H must have order cd, where $c|(n/d)$. Consequently

$$\sum_{\substack{H^*-\text{cyclic} \\ H\subseteq H^*\subseteq G}} \mu([H^* : H]) = \sum_{c|(n/d)} \mu([H^* : H]) = \begin{cases} 1 & \text{if } n/d = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $a(H) = 1$ if $H = G$ and $a(H) = 0$ otherwise. We conclude that a cyclic group has no non-trivial Brauer-Kuroda relations, as $E^G = \mathbb{Q}$, giving us only that

$$\zeta_{\mathbb{Q}}(s) = \zeta_{\mathbb{Q}}(s).$$

Example 2. Let $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. It has one cyclic subgroup $\langle -1\rangle$ of order 2, and three cyclic subgroups $\langle i\rangle, \langle j\rangle, \langle k\rangle$ of order 4. As such,

$$a(\{1\}) = \frac{1}{8}\left(\mu(1) + \mu(2) + 3\mu(4)\right) = 0$$

$$a(\langle -1\rangle) = \frac{1}{4}\left(\mu(1) + 3\mu(2)\right) = -\frac{1}{2}$$

$$a(\langle i\rangle) = a(\langle j\rangle) = a(\langle k\rangle) = \frac{1}{2}\left(\mu(1)\right) = \frac{1}{2}.$$

The Brauer-Kuroda relation therefore takes the form

$$\zeta_{E^{\langle -1\rangle}}(s)\zeta_{\mathbb{Q}}(s)^2 = \zeta_{E^{\langle i\rangle}}(s)\zeta_{E^{\langle j\rangle}}(s)\zeta_{E^{\langle k\rangle}}(s). \tag{2.6}$$

Since $a(\{1\}) = 0$, the Dedekind zeta funciton $\zeta_E(s)$ of the field E does not appear in this relation, so that we may not, in fact, use its subfields to calculate $\zeta_E(s)$.

Example 3. Let $G = D_{2p} = \langle r, s \mid r^p = s^2 = 1, rs = sr^{-1}\rangle$ be the dihedral group of order $2p$, where p is an odd prime. This group is non-abelian, and contains only one unique normal subgroup, $H_p = \langle r\rangle$, the unique subgroup of order p. For $i = 1, \ldots, p$ let $H_{2,i}$ denote the p subgroups of order 2. Clearly, neither H_p nor any of the p

subgroups $H_{2,i}$ are contained in a cyclic subgroup H^* of G, and hence

$$a(H_p) = \frac{1}{2}$$

$$a(H_{2,i}) = \frac{1}{p} \quad \text{for} \quad i = 1, \dots p$$

$$a(\{1\}) = \frac{1}{2p}(p\mu(2) + \mu(p) + \mu(1)) = -\frac{1}{2}.$$

The $H_{2,i}$ are pair-wise conjugate subgroups, and give rise to isomorphic fixed fields $E^{H_{2,1}}$. As such, they have the same zeta function, so that our Brauer-Kuroda relation reads

$$\zeta_E(s)\zeta_{\mathbb{Q}}(s)^2 = \zeta_{E^{H_p}}(s)\zeta_{E^{H_2}}(s)^2, \tag{2.7}$$

where H_2 is any one of the subgroups of order 2.

Example 4. Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, where p is a prime. Denote by H_i for $i = 1, \dots, p+1$, the $p+1$ cyclic subgroups of order p. Including the trivial subgroup $\{1\}$, these are all of the cyclic subgroups of G, and, as they are all maximal, we have

$$a(H_i) = \frac{1}{p}\left(\mu(1)\right) = \frac{1}{p} \quad \text{for} \quad i = 1, \dots, p+1$$

$$a(\{1\}) = \frac{1}{p^2}\left((p+1)\mu(p) + \mu(1)\right) = -\frac{1}{p}.$$

Letting $E_i = E^{H_i}$ for $i = 1, \dots, p+1$, it follows that

$$\zeta_{\mathbb{Q}}(s) = \left(\prod_{i=1}^{p+1} \zeta_{E_i}(s)\right)^{1/p} \zeta_E(s)^{-1/p},$$

and hence the Brauer-Kuroda relation for $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ reads

$$\zeta_{\mathbb{Q}}(s)^p \zeta_E(s) = \prod_{i=1}^{p+1} \zeta_{E_i}(s). \tag{2.8}$$

Example 5. Consider $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, the multicyclic group of order p^3. Let H_i, $i = 1, \dots, p^2 + p + 1$ denote the $p^2 + p + 1$ maximal cyclic subgroups of order

p. Including the trivial subgroup $\{1\}$, we then have:

$$a(H_i) = \frac{1}{p^2}\left(\mu(1)\right) = \frac{1}{p^2} \quad \text{for} \quad i = 1, \ldots, p^2+p+1$$

$$a(\{1\}) = \frac{1}{p^3}\left((p^2+p+1)\mu(p) + \mu(1)\right) = -\left(\frac{p+1}{p^2}\right).$$

Letting $E_i = E^{H_i}$ for $i = 1, \ldots, p^2+p+1$, it follows that

$$\zeta_{\mathbb{Q}}(s) = \left(\prod_{i=1}^{p^2+p+1} \zeta_{E_i}(s)\right)^{1/p^2} \zeta_E(s)^{(-p-1)/p^2},$$

which simplifies to

$$\zeta_{\mathbb{Q}}(s)^{p^2}\zeta_E(s)^{p+1} = \prod_{i=1}^{p^2+p+1} \zeta_{E_i}(s). \tag{2.9}$$

By (2.8), this Brauer-Kuroda relation can be simplified further:

$$\zeta_{\mathbb{Q}}(s)^{p^2}\zeta_E(s)^{p+1} = \prod_{i=1}^{p^2+p+1} \zeta_{\mathbb{Q}}(s)^{-p} \prod_{j=1}^{p+1} \zeta_{K_j}(s),$$

where each K_j, for $j = 1, \ldots, p+1$, corresponds to any one of the $p+1$ subfields of E_i, $i = 1, \ldots, p^2+p+1$. These are the subfields of E fixed by all subgroups of G of order p^2, so that they are of degree p over $\mathbb{Q}$. Rearranging, we have

$$\zeta_{\mathbb{Q}}(s)^{p^2}\zeta_E(s)^{p+1} = \zeta_{\mathbb{Q}}(s)^{-p(p^2+p+1)} \left(\prod_{i=1}^{p^2+p+1} \zeta_{K_i}(s)\right)^{p+1},$$

giving us finally

$$\zeta_{\mathbb{Q}}(s)^{p^2+p}\zeta_E(s) = \prod_{i=1}^{p^2+p+1} \zeta_{K_i}(s), \tag{2.10}$$

where K_i ranges over all degree p extensions over $\mathbb{Q}$.

Example 6. Note that the simplifying techniques in Example 5 can be used more generally to calculate Brauer relations for multicyclic groups $G = (\mathbb{Z}/p\mathbb{Z})^n$ of order p^n for any $n \in \mathbb{N}$. Consider then, such a multicyclic group G. Here, we calculate the values $a(H)$ as before, where our only cyclic subgroups are those of order p, as well

as $\{1\}$. There are $p^{n-1}+p^{n-2}+\cdots+p+1$ such subgroups, which we denote by H_i, for $i=1,\ldots,p^{n-1}+\cdots+p+1$. Aside from $\{1\}$, these subgroups are all maximal, and therefore

$$a(H_i)=\frac{1}{p^{n-1}}\left(\mu(1)\right)=\frac{1}{p^{n-1}} \quad \text{for} \quad i=1,\ldots,(p^{n-1}+\cdots p+1)$$

$$a(\{1\})=\frac{1}{p^n}\left((p^{n-1}+\cdots+p+1)\mu(p)+\mu(1)\right)=-\frac{p^{n-2}+\cdots+p+1}{p^{n-1}}.$$

Letting $E_i=E^{H_i}$ for $i=1,\ldots,(p^{n-1}+\cdots+p+1)$, it follows that

$$\zeta_{\mathbb{Q}}(s)^{p^{n-1}}\zeta_E(s)^{p^{n-2}+\cdots+p+1}=\prod_{i=1}^{p^{n-1}+\cdots+p+1}\zeta_{E_i}(s).$$

Using the Brauer-Kuroda relation for $(\mathbb{Z}/p\mathbb{Z})^{n-1}$, it follows that

$$\zeta_{\mathbb{Q}}(s)^{p^{n-1}+\cdots+p}\zeta_E(s)=\prod_{j=1}^{p^{n-1}+\cdots+p+1}\zeta_{K_j}(s), \tag{2.11}$$

where the product runs through all subfields K_j of E of degree p over $\mathbb{Q}$.

Example 7. Let $G=A_4$, the alternating group of order 12. This is the group of permutations on $\{1,2,3,4\}$, consisting of a total of 8 cyclic subgroups, all maximal, save for the trivial group $\{1\}$. Namely, for $j=1,\ldots,4$, let $H_{3,j}$ denote the cyclic subgroups of order 3, and, for $i=1,2,3$, let $H_{2,i}$ denote the cyclic subgroups of order 2. We have

$$a(\{1\})=\frac{1}{12}\left(3\mu(2)+4\mu(3)+\mu(1)\right)=-\frac{1}{2}$$

$$a(H_{2,i})=\frac{1}{6}\left(\mu(1)\right)=\frac{1}{6} \quad \text{for} \quad i=1,2,3$$

$$a(H_{3,j})=\frac{1}{4}\left(\mu(1)\right)=\frac{1}{4} \quad \text{for} \quad j=1,\ldots,4.$$

Denote by $\zeta_{E\sigma}$ the zeta function of the subfield of E fixed by the automorphism σ, and observe that the $H_{2,i}$ and the $H_{3,j}$ are all pair-wise conjugate subgroups, respectively. Hence, as in the dihedral case, they have the same zeta functions, respectively, and therefore

$$\zeta_E(s)\zeta_{\mathbb{Q}}(s)^2=\zeta_{E(12)(34)}(s)\zeta_{E(234)}(s)^2. \tag{2.12}$$

As illustrated by Example 2, there are cases in which $\zeta_E(s)$ does not appear in the Brauer-Kuroda relation. Namely, those in which $a(\{1\}) = 0$. For all other cases however, the clear implication of the Brauer-Kuroda relations is that one has only to calculate the zeta function values for certain subgroups in order to determine the zeta function of E, the top field. In the case of the multicyclic groups $(\mathbb{Z}/p\mathbb{Z})^n$ in particular, we have seen that $\zeta_E(s)$ depends only on the degree p subextensions over $\mathbb{Q}$.

In the sections to follow, we will evaluate the Dedekind zeta function of various number fields at different values of s. Generally speaking, these values are difficult to calculate, even for computational software, and the complexity of the computation increases with the degree of the number field in question. These Brauer-Kuroda relations, however, though they may not hold for all Galois groups $G = Gal(E/F)$, can easily be used to simplify our calculations, as we will see.

Chapter 3

Cohomology

In 1971, Lichtenbaum gave a conjectural interpretation of the values of the Dedekind zeta function of a number field E at negative odd integers. He suggested a generalization of Dirichlet's class number formula in terms of algebraic K-theory, which was later proved true for abelian fields.

Following Lichtenbaum's Conjecture, Bartel, de Smit [4] proved certain relations on higher K-groups of rings of integers arising from Brauer-Kuroda relations on the Dedekind zeta function, evaluated at negative odd integers. However, in light of the K-theoretic approach, there is a notable discrepancy at the 2-torsion, which may be rectified by replacing K-groups by motivic cohomology groups.

In this chapter, we first review some necessary cohomological background needed to subsequently discuss the Lichtenbaum Conjecture, and finally, we present a reformulation of this conjecture in terms of motivic cohomology.

The cohomology theory presented here is primarily taken from [6], [26] and [29].

3.1 Cohomology of Groups

3.1.1 G-Modules

Let G be a group and consider a G-module A. Since an abelian group is the same as a module over $\mathbb{Z}$, it is easy to see that a G-module is the same as a module over the integral group ring $\Lambda = \mathbb{Z}G$ of G, so that we may think of A as a Λ-module. Given two such G-modules A and B, a map $A \to B$ is called a G-homomorphism if it is a homomorphism of Λ-modules, where the set of all such G-homomorphisms forms a group, denoted by $Hom_G(A, B)$.

Writing $g \cdot a$ for the action of the element $g \in G$ on the element $a \in A$, we let $A^G = \{a \in A \mid g \cdot a = a \text{ for all } g \in G\}$ be the subset of elements of A which are invariant under the action of G. This is an abelian group which depends functorially on A, and is in fact the largest submodule of A on which G acts trivially. In particular,

$$A^G \cong Hom_G(\mathbb{Z}, A),$$

where $\mathbb{Z}$ is regarded as a G-module on which G acts trivially.

We briefly recall that, given a fixed R-module D, where R is a commutative ring, the map $Hom_R(D, -)$ is a covariant functor from the category of R-modules to the category of abelian groups. In particular, this functor is left exact. As such, it follows easily that a short exact sequence of G-modules

$$0 \to A \to B \to C \to 0$$

induces an exact sequence of abelian groups

$$0 \to A^G \to B^G \to C^G,$$

which does not, in general, terminate with a surjective map. As we shall see in the next section, this sequence extends in a natural way to a long exact sequence in cohomology.

3.1.2 Definition of Cohomology

As before, consider $\mathbb{Z}$ as a G-module under trivial G action. A projective resolution of $\mathbb{Z}$ is an exact sequence

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

such that each P_i is a projective G-module. In other words, each $Hom_G(P_i, -)$ is an exact functor.

Given such a resolution, we may form a related sequence by taking G-module homomorphisms from the projective resolution into A, keeping in mind that this reverses the direction of the homomorphisms. This yields the sequence

$$\begin{aligned} 0 \to Hom_G(\mathbb{Z}, A) \xrightarrow{\epsilon} Hom_G(P_0, A) \xrightarrow{d_1} Hom_G(P_1, A) \xrightarrow{d_2} \cdots \\ \cdots \xrightarrow{d_{n-1}} Hom_G(P_{n-1}, A) \xrightarrow{d_n} Hom_G(P_n, A) \xrightarrow{d_{n+1}} \cdots , \end{aligned} \tag{3.1}$$

which is not necessarily exact, but one can show that the composition of any two successive maps is zero, i.e. $d_{n+1} \circ d_n = 0$ for all n. As such, we call this associated sequence a *complex*.

In this case, as $im(d_n) \subseteq ker(d_{n+1})$ for all n, we can consider the quotient

$$H^n(G, A) = ker(d_{n+1})/im(d_n),$$

defined for all $n \geq 1$, with $H^0(G, A) = ker(d_1)$. The group $H^n(G, A)$ is called the nth *cohomology group* of G with coefficients in A. These groups do not depend on the choice of projective resolution, and consequently, $H^n(G, A)$ defines a contravariant functor from the category of G-modules to the category of abelian groups. It follows that the first four terms of the sequence

$$0 \to Hom_G(\mathbb{Z}, A) \xrightarrow{\epsilon} Hom_G(P_0, A) \xrightarrow{d_1} Hom_G(P_1, A)$$

are exact so that $ker(d_1) = im(\epsilon) \cong Hom_G(\mathbb{Z}, A) \cong A^G$. In particular, this gives $H^0(G, A) \cong A^G$.

We note that the groups $H^n(G, A)$ are also the cohomology groups of the cochain complex 3.1 obtained by replacing the term $Hom_G(\mathbb{Z}, A)$ with zero. In either case, it is clear that both cochain complexes are exact sequences if and only if $H^n(G, A) = 0$

for all n. Hence, the nth cohomology group measures the failure of exactness of the complex at the nth stage.

Finally, returning to our short exact sequence of G-modules

$$0 \to A \to B \to C \to 0,$$

an important result in cohomology theory states that we can define boundary homomorphisms

$$\delta_n : H^n(G, C) \to H^{n+1}(G, A),$$

thereby obtaining a long exact sequence of abelian groups:

$$0 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \xrightarrow{\delta_0} H^1(G, A) \to H^1(G, B) \to H^1(G, C) \xrightarrow{\delta_1} \cdots$$

$$\cdots \xrightarrow{\delta_{n-1}} H^n(G, A) \to H^n(G, B) \to H^n(G, C) \xrightarrow{\delta_n} \cdots$$

Recall that $H^0(G, A) \cong A^G$, so that the cohomology groups $H^n(G, A)$ satisfy the required properties to be a cohomological extension of the functor $A \to A^G$.

3.2 Galois Cohomology

3.2.1 Definition of Galois Cohomology

An important application of group cohomology arises when the group G is taken to be the Galois group of a field extension K/F. Fortunately, the modifications necessary to define the cohomology groups in this context are relatively minor and apply to arbitrary profinite groups.

Of particular importance is the absolute Galois group $G_F = Gal(\bar{F}/F)$ where $K = \bar{F}$ is the separable closure of F. This is a profinite group, meaning that $G_F = Gal(\bar{F}/F)$ is the inverse limit $\varprojlim Gal(L/F)$ of the Galois groups of the finite extensions L of F contained in $\bar{F}$. Further, G_F is a compact topological group with respect to the Krull topology.

A *discrete* G_F-module is a G_F-module A with the discrete topology such that

the action of G_F on A is continuous. i.e. the map $G_F \times A \to A$ taking (g, a) to $g \cdot a$ is continuous. Given such a discrete G_F-module A, the cohomology groups $H^i(G_F, A)$ computed using continuous cochains are called *Galois cohomology groups*. We will denote these simply as $H^i(F, A)$.

3.2.2 Tate Twists

Let F be a number field and let μ denote the group of roots of unity. This is a G_F-module in a natural way. Fix a prime p and let $\mu_{p^m}^{\otimes n}$ be the n-fold tensor product of this group, where μ_{p^m} is the group of p^m-th roots of unity. Note that this n-fold tensor product $\mu_{p^m}^{\otimes n}$ of discrete G_F-modules is itself a discrete G_F-module under the diagonal G-action defined by $g(\zeta \otimes \cdots \otimes \zeta) = g(\zeta) \otimes \cdots \otimes g(\zeta)$. We define the projective limit

$$\mathbb{Z}_p(n) = \varprojlim \mu_{p^m}^{\otimes n} \tag{3.2}$$

and inductive limit

$$\mathbb{Q}_p/\mathbb{Z}_p(n) = \mu_{p^\infty}^{\otimes n} = \varinjlim \mu_{p^m}^{\otimes n}. \tag{3.3}$$

For $n \in \mathbb{Z},\ n \neq 0$, we introduce the numbers

$$w_n(F)(p) := \max\{p^m \mid Gal(F(\mu_{p^m})/F) \text{ has exponent } n\}. \tag{3.4}$$

In particular, $w_1(F)(p)$ is the number of p-power roots of unity in F.

In terms of Galois cohomology, $w_n(F)(p)$ is simply the largest prime power p^m such that $Gal(\bar{F}/F)$ acts trivially on the n-fold tensor product $\mu_{p^m}^{\otimes n}$. Explicitly, this means

$$w_n(F)(p) = |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n))|, \tag{3.5}$$

where

$$H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong \varinjlim H^0(F, \mu_{p^m}^{\otimes n}). \tag{3.6}$$

To see this, note that, by definition, $H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is the subset of elements of $\mathbb{Q}_p/\mathbb{Z}_p(n)$ which are invariant under the action of $G_F = Gal(F(\mu_{p^\infty})/F)$, where $F(\mu_{p^\infty})/F$ is obtained by adjoining to F all roots of unity of p-power order. If $g \in G_F$, we have that $g(\zeta) = \zeta^a$ for some integer a, where $\zeta \in \mu_{p^m}$. Hence, in looking at the

action of G_F on the tensor product, we obtain

$$g(\zeta^{\otimes n}) = g(\zeta) \otimes \cdots \otimes g(\zeta)$$

$$= \zeta^a \otimes \cdots \otimes \zeta^a$$

$$= (\zeta^{\otimes n})^{a^n},$$

so that the action of g on $\mu_{p^m}^{\otimes n}$ is identical to the action of g^n on μ_{p^m}. More precisely, this means

$$H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \{\zeta \in \mu_{p^\infty} \mid g^n\zeta = \zeta \ \text{ for all } g \in G_F\},$$

and therefore

$$|H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n))| = \max\{p^m \mid g^n = 1 \ \text{ for all } g \in Gal(F(\mu_{p^m})/F)\} = w_n(F)(p)$$

as claimed.

We briefly remark that the group $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to the p-power torsion subgroup of $\mathbb{Q}/\mathbb{Z}$ so that $\mathbb{Q}/\mathbb{Z} \cong \oplus_p \mathbb{Q}_p/\mathbb{Z}_p$. Following this, we have that

$$H^0(F, \mathbb{Q}/\mathbb{Z}(n)) = \prod_p H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n)), \tag{3.7}$$

where p runs through all primes.

3.3 Étale Cohomology

In our present situation, we can use a description of étale cohomology in terms of Galois cohomology, where the scheme will be the affine scheme $spec(R)$ for a certain Dedekind ring R, and the sheaves will consist of roots of unity.

Suppose E is a number field and p a fixed prime number. Let $o'_E = o_E[\frac{1}{p}]$, where o_E is the ring of integers of E. This is the ring of S integers of E, where S is the set of primes above p as well as set of infinite primes. Let E_S denote the maximal algebraic extension of E which is unramified outside primes above p and infinite primes. We denote its corresponding Galois group by $G_S = Gal(E_S/E)$.

As shown by Milne (exercise 3.16 in [22]), the étale cohomology groups $H^i_{ét}(spec(o'_E), \mu_{p^m}^{\otimes n})$ of $spec(o'_E)$ coincide with the Galois cohomology groups $H^i(G_S, \mu_{p^m}^{\otimes n})$, and will be denoted simply by $H^i_{ét}(o'_E, \mu_{p^m}^{\otimes n})$.

Additionally, let

$$H^i_{ét}(o'_E, \mathbb{Z}_p(n)) = \varprojlim H^i_{ét}(o'_E, \mu_{p^m}^{\otimes n}) \tag{3.8}$$

denote the p-adic étale cohomology groups, and set

$$H^i_{ét}(o'_E, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \varinjlim H^i_{ét}(o'_E, \mu_{p^m}^{\otimes n}). \tag{3.9}$$

Finally, we have

$$H^2(o_E, \mathbb{Z}(n)) = \prod_p H^2_{ét}(o'_E, \mathbb{Z}_p(n)), \tag{3.10}$$

whose order, denoted by $h_n(E)$, is called the *higher class number* of E.

3.4 Motivic Cohomology

The p-adic étale cohomology groups are defined for a single prime p, but we would like to view them as the p-part of a 'global' cohomology. Two candidates for this 'global' cohomology are Algebraic K-Theory and Motivic Cohomology.

Consider first algebraic K-theory. The Quillen-Lichtenbaum Conjecture predicts that there are isomorphisms

$$K_{2n-i}(o_E) \otimes \mathbb{Z}_p \cong H^i_{ét}(o'_E, \mathbb{Z}_p(n)),$$

for $n \geq 2, i = 1, 2$, unless $p = 2$ and E is a number field with a real embedding. In the exceptional case that E has such a real embedding, the relation between higher K-groups of o_F (tensored by $\mathbb{Z}_2$) and dyadic étale cohomology groups is quite complicated for certain indices and need not be an isomorphism in these particular cases ([27]).

In general, the Quillen-Lichtenbaum Conjecture is a consequence of the Bloch-Kato Conjecture, which implies that there are isomorphisms

$$K_{2n-i}(o_E) \cong H^i_{\mathcal{M}}(o_E, \mathbb{Z}(n))$$

up to 2-torsion for all $n \geq 2$ and $i = 1, 2$. On the other hand, the Bloch-Kato Conjecture was proven by Rost and Voevodsky, giving validity to the above results in this section.

While the 2-torsion is different, it is motivic cohomology which has the property that

$$H^i_{\mathcal{M}}(o_E, \mathbb{Z}(n)) \otimes \mathbb{Z}_p \cong H^i_{\acute{e}t}(o'_E, \mathbb{Z}_p(n)) \tag{3.11}$$

for *all* primes p, so that indeed the motivic cohomology groups of o_E behave much better than the algebraic K-groups.

We will be particularly interested in the 2nd motivic cohomology groups $H^2_{\mathcal{M}}(o_E, \mathbb{Z}(n))$ of o_E for $n \geq 2$. In this case, as a consequence of 3.11, we obtain an isomorphism between finite groups

$$H^2_{\mathcal{M}}(o_E, \mathbb{Z}(n)) \cong \bigoplus_p H^2_{\acute{e}t}(o'_E, \mathbb{Z}_p(n)).$$

3.5 Motivic Class Number Formulae

At $s = 0$, the Analytic Class Number Formula, discussed in Section 1.2, illustrates a strong connection between the special value of the Dedekind zeta function of a number field E and the class number of E. Consider now the special values at $1 - n$ for $n \geq 2$.

We have seen previously that the functional equation for the Dedekind zeta function implies that the order of vanishing is related to the number of real and complex embeddings of E. Namely, by 1.4, the value $\zeta_E(1-n)$ is non-zero precisely when $n \geq 2$ is even and E is a totally real number field. Hence, in this case, $\zeta^*_E(1-n) = \zeta_E(1-n)$.

As a consequence of the Main Conjecture in Iwasawa theory, this value is given by the following theorem from [15]

Theorem 3.1. *(Wiles) Let E be a totally real number field with ring of integers o_E. For all even integers $n \geq 2$,*

$$\zeta_E(1-n) = \pm \frac{h_n(E)}{w_n(E)} \tag{3.12}$$

up to a power of two. If E is abelian over $\mathbb{Q}$, then the formula also gives the correct power of 2.

For odd primes p, the Main Conjecture was proven by Wiles ([36]) for arbitrary totally real number fields E, and for the prime 2 in the case $E = \mathbb{Q}$. In the remainder of this paper, we assume that the Main Conjecture for $p = 2$ is valid in general, in which case the 2-primary part of the Theorem 3.1 holds as well.

More generally, suppose E is an arbitrary number field, and consider again the special values of the Dedekind zeta function at negative odd integers. In this case, Lichtenbaum conjectured that the correct generalization of Dirichlet's Class Number Formula should read:

Conjecture 1. *(Lichtenbaum) For all $n \geq 2$:*

$$\zeta_E^*(1-n) = \pm \frac{|K_{2n-2}(o_E)|}{|K_{2n-1}(o_E)_{tors}|} \cdot R_n^B(E), \tag{3.13}$$

up to powers of 2*, where $R_n^B(E)$ denotes the Borel regulator.*

The following main result shows that Lichtenbaum's Conjecture is true for abelian fields ([17]):

Theorem 3.2. *Let E be an abelian number field. Then for all $n \geq 2$,*

$$\zeta_E^*(1-n) = \pm \frac{h_n(E)}{w_n(E)} \cdot R_n(E), \tag{3.14}$$

up to powers of 2*, where $R_n(E)$ denotes the Beilinson regulator.*

As Lichtenbaum's Conjecture includes a difference at the 2-torsion, in order to obtain a formulation including the exact power of 2, we must consider motivic cohomology in place of algebraic K-theory. More precisely, in [15], it is shown that the 'correct' formulation of the Lichtenbaum Conjecture should read:

Conjecture 2.

$$\zeta_E^*(1-n) = \pm \frac{|H^2_{\mathcal{M}}(E, \mathbb{Z}(n))|}{|H^1_{\mathcal{M}}(E, \mathbb{Z}(n))_{tors}|} \cdot R_n^{\mathcal{M}}(E), \tag{3.15}$$

where $R_n^{\mathcal{M}}$ is the motivic regulator.

Chapter 4

Main Result

Upon introducing Brauer-Kuroda relations, in his original paper, Brauer also proved that these relations hold for all odd p-power roots of unity of the relevant subfields. In our previous notation, this is a result on the values $w_1(E)$, indicating, by Theorem 1.1, that the class number of a field is related to those of certain subfields up to 2-torsion.

Bartel and De Smit [4] later improved this result, generalizing it to arbitrary $w_n(E)$ via algebraic K-theory, and showing that the relation holds for $w_n(E)$ for all n up to 2-torsion. Consequently, for n even, given a Galois extension of totally real number fields E/F with Galois group G, it follows by Theorem 3.1 that the Brauer-Kuroda relations hold on higher class numbers, up to a power of 2.

In this chapter, we establish another improvement, describing the fields E for which $w_n(E)$ cancel in the Brauer-Kuroda relations, without any extra factors of 2.

4.1 Walter's Lemma

Let E/F be a normal extension of algebraic number fields with Galois group $G = Gal(E/F)$, and let E^H denote the subfield of E fixed by the subgroup H of G. Letting $w_1(K)$ denote the number of roots of unity of the field K and $w_1(K)(p)$ its

corresponding p-part, Brauer proved

Theorem 4.1. *(Brauer) If*

$$\sum_H a(H) Ind_H^G(\mathbf{1}_H) = 0$$

denotes a Brauer-Kuroda relation on G, then

$$\prod_H w_1(E^H)^{a(H)} = \prod_H w_1(E^H)(2)^{a(H)}.$$

Suppose also that (ζ_{2^m}) is the group of 2-power roots of unity in E and $F(\zeta_{2^m})/F$ is cyclic. Then

$$\prod_H w_1(E^H)(2)^{a(H)} = 1.$$

Instead of reciting Brauer's proof for the above theorem, we follow Walter's proof, which is shorter and more natural. In doing so, we require the following lemma, presented in [34]:

Lemma 4.2. *Suppose M is a finite $\mathbb{Z}G$-module fixed by a normal subgroup N over which G is cyclic. If*

$$\sum_H a(H) Ind_H^G(\mathbf{1}_H) = \sum_H b(H) Ind_H^G(\mathbf{1}_H),$$

where $a(H)$ and $b(H)$ are non-negative integers, then there is a trivial group isomorphism

$$\oplus_H \oplus_{i=1}^{a(H)} (M^H)^{(i)} \cong \oplus_H \oplus_{j=1}^{b(H)} (M^H)^{(j)}$$

for $M^{(j)} \cong M$.

Here, following our previous notation, A^H denotes the submodule fixed under a subgroup H of G, where A is a G-module.

We are now in position to prove Theorem 4.1:

Proof. Let (ζ_{p^m}) denote the group of p-power roots of unity in E. This is clearly a G-module. With the possible exception of $p = 2$, $F(\zeta_{p^m})/F$ is cyclic, and, as $N = Gal(E/F(\zeta_{p^m}))$ is a normal subgroup of G, the theorem is a direct consequence of Lemma 4.2 on taking orders. □

As an easy example where $w_1(E)$ fails to cancel in a Brauer-Kuroda relation, consider the field extension $\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}$. Here, $G = Gal(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and hence the associated relation reads

$$\zeta_{\mathbb{Q}(i,\sqrt{2})}(s)\zeta_{\mathbb{Q}}(s)^2 = \zeta_{\mathbb{Q}(\sqrt{2})}(s)\zeta_{\mathbb{Q}(\sqrt{-2})}(s)\zeta_{\mathbb{Q}(i)}(s).$$

Looking at $w_1(F)$, we note that this relation gives $\prod_H w_1(F^H)^{a(H)} = 2$, in compliance with Theorem 4.1.

4.2 Brauer-Kuroda Relations for $w_n(E)$

More generally, consider now the values $w_n(E) = |H^0(E, \mathbb{Q}/\mathbb{Z}(n))|$, for arbitrary n, keeping in mind that $w_1(E)$ denotes the number of roots of unity of E. Analogous to Brauer's result, the main theorem of this thesis states:

Proposition 4.3. *Given a Brauer-Kuroda relation on G*

$$\sum_H a(H) Ind_H^G = 0,$$

it follows that

$$\prod_H w_n(E^H)^{a(H)} = 1$$

for all n even. For n odd,

$$\prod_H w_n(E^H)^{a(H)} = \prod_H w_1(E^H)(2)^{a(H)}.$$

Proof. We proceed by showing that $H^0(E, \mathbb{Z}_p/\mathbb{Q}_p(n))$ is fixed by a normal subgroup N over which G is cyclic. The result will then follow by Lemma 4.2 upon taking orders.

Recall that

$$w_n(E)(p) = \max\{p^m \mid Gal(E(\mu_{2^m})/E) \text{ has exponent } n\},$$

and let $M = H^0(E, \mathbb{Z}_p/\mathbb{Q}_p(n))$ so that $|M| = w_n(E)(p)$. For $n \geq 1$ even, suppose that $w_n(E)(p) = p^k$ for some $k \in \mathbb{N}$. We then have the following scenario:

$$\begin{tikzcd}
 & E(\zeta_{p^k}) \arrow[dl, dash] \arrow[dd, dash] \\
E \arrow[dd, dash] & \\
 & F(\zeta_{p^k}) \arrow[dl, dash] \\
E \cap F(\zeta_{p^k}) \arrow[d, dash] & \\
F &
\end{tikzcd}$$

Since $E \cap F(\zeta_{p^k})$ and E contain the same p-power roots of unity, we first observe that $|H^0(E \cap F(\zeta_{p^k}), \mathbb{Z}_p/\mathbb{Q}_p(n))| = |M|$. As two finite abelian groups with the same number of elements of each order are isomorphic, it follows that $M \cong H^0(E \cap F(\zeta_{p^k}), \mathbb{Z}_p/\mathbb{Q}_p(n))$. Here, $H^0(E \cap F(\zeta_{p^k}), \mathbb{Z}_p/\mathbb{Q}_p(n))$ is invariant under $N = Gal(E/E \cap F(\zeta_{p^k}))$, and therefore M is fixed by N as well.

We now have

$$\begin{tikzcd}
 & E(\zeta_{p^k}) \arrow[dl, dash] \arrow[dr, dash] & \\
E \arrow[dr, dash, "N"] \arrow[ddr, dash] & & F(\zeta_{p^k}) \arrow[dl, dash] \arrow[ddl, dash] \\
 & E \cap F(\zeta_{p^k}) \arrow[d, dash] & \\
 & F &
\end{tikzcd}$$

To see that N is a normal subgroup of $G = Gal(E/F)$, note simply that $E \cap F(\zeta_{p^k})$ is Galois over F. Finally, note that with the possible exception of $p = 2$, $E(\zeta_{p^k})/E$ and $F(\zeta_{p^k})/F$ are cyclic extensions. In this case, as $Gal(E(\zeta_{p^k})/E) \cong Gal(F(\zeta_{p^k})/E \cap F(\zeta_{p^k}))$, necessarily, $E \cap F(\zeta_{p^k})/F$ must also be a cyclic extension, and hence, by Walters's Lemma, we are reduced to the 2-torsion subgroup of $w_n(E)$.

Consider $p = 2$ and suppose $Gal(E(\mu_{2^k})/E) = \mathbb{Z}/2^m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for some $m \in \mathbb{N}$.

Let $\Delta = \mathbb{Z}/2^m\mathbb{Z}$ so that $Gal(E(\mu_{2^k})/E(\mu_{2^k})^\Delta) = \Delta$. Since n is even,

$$|H^0(E, \mathbb{Z}_2/\mathbb{Q}_2(n))| = \max\{2^m \mid Gal(E(\mu_{2^m})/E) \text{ has exponent } n\}$$

$$= \max\{2^m \mid Gal(E(\mu_{2^m})^\Delta/E) \text{ has exponent } n\},$$

and hence we can reduce the scenario to

$$\begin{tikzcd}
 & E(\zeta_{p^k})^\Delta \arrow[dl, dash] \arrow[dr, dash] & \\
E \arrow[dr, dash] \arrow[ddr, dash] & & F(\zeta_{p^k})^\Delta \arrow[dl, dash] \arrow[ddl, dash] \\
 & E \cap F(\zeta_{p^k})^\Delta \arrow[d, dash] & \\
 & F &
\end{tikzcd}$$

The same argument as above shows that $H^0(E, \mathbb{Z}_2/\mathbb{Q}_2(n)) \cong H^0(E \cap F(\zeta_{2^k})^\Delta, \mathbb{Z}_2/\mathbb{Q}_2(n))$, where $H^0(E \cap F(\zeta_{2^k})^\Delta, \mathbb{Z}_2/\mathbb{Q}_2(n))$ is invariant under $\tilde{N} = Gal(E/E \cap F(\zeta_{2^k})^\Delta)$. Similarly, that $\tilde{N}$ is a normal subgroup of G also follows as above, this time because $E \cap F(\zeta_{2^k})^\Delta$ is Galois over F.

It remains to show that $E \cap F(\zeta_{2^k})^\Delta$ is cyclic over F. However, we use the same argument as for p odd, since taking the fixed fields $E(\zeta_{2^k})^\Delta$ and $F(\zeta_{2^k})^\Delta$ immediately implies that $E(\zeta_{2^k})^\Delta/E$ and $F(\zeta_{2^k})^\Delta/F$ are cyclic extensions. The Proposition then follows by Walter's Lemma.

Suppose now that n is odd. In this case, if $n > 1$, adjoining 2-power roots of unity never results in an odd degree extension and thus $w_n(E^H)(2) = w_1(E^H)(2)$.

$\square$

Putting together Proposition 4.3 and Theorem 4.1, we observe that

$$\prod_H w_n(E^H)^{a(H)} = 1$$

in a Brauer-Kuroda relation for all n when E/F is an extension of totally real fields. This follows trivially as $w_1(E^H)(2) = 2$ for all subfields E^H of E. Additionally, if $F(\zeta_{2^m})/F$ is cyclic, where (ζ_{2^m}) is the group of 2-power roots of unity in E, then

$$\prod_H w_n(E^H)(2)^{a(H)} = 1$$

for all n odd.

Chapter 5

Computations for Higher Class Numbers

In previous chapters, we have seen that Brauer-Kuroda relations on a number field E depend entirely on the Galois group $G = Gal(E/F)$ and not on the fields E and F themselves. As a result, these relations can be applied to a very large set of number field extensions E/F, even without changing the Galois group. In this chapter, we focus our attention to totally real field extensions with multicyclic Galois groups. Using our main theorem in Section 4.2, we give explicit examples of such fields, calculating their higher class numbers $h_n(E)$ in the case that n is even.

All computations involving the Dedekind zeta function presented here were performed in Sage ([30]).

5.1 Multicyclic Fields

Using 2.11, we calculate $\zeta_E(s)$ at $s = -1$ and $s = -3$ of the totally real number field E whose Galois group $G = Gal(E/\mathbb{Q})$ is multicyclic. Namely, we consider the field $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11})$ over $\mathbb{Q}$, and several of its subfields, letting E vary as any one of these subfields.

To do so, we first make note of Table 5.1, where we compute the values of the zeta function for all quadratic subfields of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11})$. As Brauer-Kuroda relations between multicyclic fields depend only on the subfields fixed by $\mathbb{Z}/2\mathbb{Z}$, we compute $\zeta_E(s)$ using this table and present the results in Tables 5.2 - 5.6. The class numbers $h_n(E)$ are then obtained by Theorem 3.1 and presented in these tables, respectively.

Remark 2. In calculating $\zeta_{E^H}(-3)$ of certain quadratic subfields E^H, Sage documents a loss of decimal digits, and gives an approximate value instead. These values are marked with an asterisks (*), and are assumed to be correct as they give integral values of $h_4(E)$. Those that do not have been omitted.

Table 5.1: Dedekind Zeta Function Values of Totally Real Quadratic Number Fields

	E^H	$\zeta_{E^H}(-1)$	$\zeta_{E^H}(-3)$
	$\mathbb{Q}$	$1/12$	$1/120$
	$\mathbb{Q}(\sqrt{2})$	$1/12$	$11/120$
	$\mathbb{Q}(\sqrt{3})$	$1/6$	$23/60$
	$\mathbb{Q}(\sqrt{5})$	$1/30$	$1/60$
	$\mathbb{Q}(\sqrt{6})$	$1/2$	$87/20$
	$\mathbb{Q}(\sqrt{7})$	$2/3$	$113/15$
	$\mathbb{Q}(\sqrt{10})$	$7/6$	$1577/60$
	$\mathbb{Q}(\sqrt{11})$	$7/6$	$2153/60$
	$\mathbb{Q}(\sqrt{14})$	$5/3$	$2503/30$
	$\mathbb{Q}(\sqrt{15})$	2	$537/5$
	$\mathbb{Q}(\sqrt{21})$	$1/3$	$77/30$
	$\mathbb{Q}(\sqrt{22})$	$23/6$	$24889/60$
	$\mathbb{Q}(\sqrt{30})$	$17/3$	$36451/30$
	$\mathbb{Q}(\sqrt{33})$	1	$141/10$
	$\mathbb{Q}(\sqrt{35})$	$19/3$	$61733/30$
(*)	$\mathbb{Q}(\sqrt{42})$	9	$7875/2$
(*)	$\mathbb{Q}(\sqrt{55})$	$46/3$	$153847/15$
(*)	$\mathbb{Q}(\sqrt{66})$	$56/3$	$288086/15$
(*)	$\mathbb{Q}(\sqrt{70})$	$67/3$	$143185/6$
	$\mathbb{Q}(\sqrt{77})$	2	$1193/5$
	$\mathbb{Q}(\sqrt{105})$	6	$4059/5$
(*)	$\mathbb{Q}(\sqrt{110})$	$103/3$	$3395981/30$
(*)	$\mathbb{Q}(\sqrt{154})$	$226/3$	$5662171/15$
(*)	$\mathbb{Q}(\sqrt{165})$	$22/3$	$52291/15$
(*)	$\mathbb{Q}(\sqrt{210})$	$310/3$	$16533469/15$
(*)	$\mathbb{Q}(\sqrt{231})$	124	$7705614/5$
(*)	$\mathbb{Q}(\sqrt{330})$	206	$26817909/5$
(*)	$\mathbb{Q}(\sqrt{385})$	$142/3$	$1163827/15$
	$\mathbb{Q}(\sqrt{462})$	$962/3$	
	$\mathbb{Q}(\sqrt{770})$	$1948/3$	
	$\mathbb{Q}(\sqrt{1155})$	$3956/3$	
	$\mathbb{Q}(\sqrt{2310})$	3792	

Table 5.2: Computations Using Multicyclic Galois Groups for the Case $n = 2$

$Gal(E/\mathbb{Q})$	E	$\zeta_E(-1)$	$h_2(E)$
$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Q}(\sqrt{2})$	$1/12$	4
$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Q}(\sqrt{2},\sqrt{3})$	1	48
	$\mathbb{Q}(\sqrt{2},\sqrt{5})$	$7/15$	112
	$\mathbb{Q}(\sqrt{2},\sqrt{7})$	$40/3$	640
	$\mathbb{Q}(\sqrt{2},\sqrt{11})$	$161/3$	2576
	$\mathbb{Q}(\sqrt{3},\sqrt{5})$	$8/5$	4800
	$\mathbb{Q}(\sqrt{3},\sqrt{7})$	$16/3$	128
	$\mathbb{Q}(\sqrt{3},\sqrt{11})$	28	672
	$\mathbb{Q}(\sqrt{5},\sqrt{7})$	$304/15$	2432
	$\mathbb{Q}(\sqrt{5},\sqrt{11})$	$1288/15$	10304
	$\mathbb{Q}(\sqrt{7},\sqrt{11})$	224	5376
$(\mathbb{Z}/2\mathbb{Z})^3$	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})$	$45696/5$	2193408
	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{7})$	69120	3317760
	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{11})$	1731072	83091456
	$\mathbb{Q}(\sqrt{2},\sqrt{5},\sqrt{7})$	$4562432/3$	364994560
	$\mathbb{Q}(\sqrt{2},\sqrt{5},\sqrt{11})$	$341742464/15$	5467879424
	$\mathbb{Q}(\sqrt{2},\sqrt{7},\sqrt{11})$	186296320	8942223360
	$\mathbb{Q}(\sqrt{3},\sqrt{5},\sqrt{7})$	$1400832/5$	33619968
	$\mathbb{Q}(\sqrt{3},\sqrt{7},\sqrt{11})$	31997952	767950848
	$\mathbb{Q}(\sqrt{3},\sqrt{5},\sqrt{11})$	$21762048/5$	522289152
	$\mathbb{Q}(\sqrt{5},\sqrt{7},\sqrt{11})$	$3558424576/5$	85402189824
$(\mathbb{Z}/2\mathbb{Z})^4$	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7})$	1148720783724380160	275692988093851238400
	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{11})$	$1304486218019012345856/5$	6261533846491259260108 8
	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{7},\sqrt{11})$	1486402845266103238656 0	7134733657277295545548 80
	$\mathbb{Q}(\sqrt{2},\sqrt{5},\sqrt{7},\sqrt{11})$	7129400431762976578469888	17110561036231143788327731 20
	$\mathbb{Q}(\sqrt{3},\sqrt{5},\sqrt{7},\sqrt{11})$	$1223135619916648025161728/5$	2935525487799955260388147 2

Table 5.3: Computations Using $G \cong (\mathbb{Z}/2\mathbb{Z})^5$ for the Case $n = 2$

E	$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11})$
$\zeta_E(-1)$	1298244775236778463130325917572779877315259048668598846357504 0
$h_2(E)$	3115787460568268311512782202174671705556621716804637231258009600

Table 5.4: Computations Using Multicyclic Galois Groups for the Case $n = 4$

$Gal(E/\mathbb{Q})$	E	$\zeta_E(-3)$	$h_4(E)$
$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Q}(\sqrt{2})$	$1/120$	4
$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Q}(\sqrt{2},\sqrt{3})$	$22011/10$	1056528
	$\mathbb{Q}(\sqrt{2},\sqrt{5})$	$17347/30$	1387760
	$\mathbb{Q}(\sqrt{2},\sqrt{7})$	$12444916/15$	398237312
	$\mathbb{Q}(\sqrt{2},\sqrt{11})$	589446187	282934169760
	$\mathbb{Q}(\sqrt{3},\sqrt{5})$	$49404/5$	11856960
	$\mathbb{Q}(\sqrt{3},\sqrt{7})$	$1600984/15$	25615744
	$\mathbb{Q}(\sqrt{3},\sqrt{11})$	$13964358/5$	670289184
	$\mathbb{Q}(\sqrt{5},\sqrt{7})$	$55806632/15$	4464530560
	$\mathbb{Q}(\sqrt{5},\sqrt{11})$	$1324930364/15$	105994429120
	$\mathbb{Q}(\sqrt{7},\sqrt{11})$	$4643900432/5$	222907220736
$(\mathbb{Z}/2\mathbb{Z})^3$	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})$	$130453708262378688/5$	62617779965941770240
	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{7})$	2899222687668384000	1391626890080824320000
	$\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{11})$	$9198854428935414443904/5$	8830900251777997866147 84
	$\mathbb{Q}(\sqrt{2},\sqrt{5},\sqrt{7})$	$1110244277440197311104 0/3$	8881954219521578488832000
	$\mathbb{Q}(\sqrt{2},\sqrt{5},\sqrt{11})$	$310820828504086473842 81152/15$	4973133256065383581484984320
	$\mathbb{Q}(\sqrt{2},\sqrt{7},\sqrt{11})$	$1153205851005444893147831296/5$	11070776169652270974219180 4416
	$\mathbb{Q}(\sqrt{3},\sqrt{5},\sqrt{7})$	$330894026433191079936/5$	7941456634396585918464 0
	$\mathbb{Q}(\sqrt{3},\sqrt{7},\sqrt{11})$	$205878169491777840460 67712/5$	9882152135605336342112 50176
	$\mathbb{Q}(\sqrt{3},\sqrt{5},\sqrt{11})$	$185324335009260550170624/5$	4447784040222253204094 9760
	$\mathbb{Q}(\sqrt{5},\sqrt{7},\sqrt{11})$	$2628135203007150424138325 1968/5$	6307524487217161017931980472320

Table 5.5: Computations Using $G \cong (\mathbb{Z}/2\mathbb{Z})^4$ for the Case $n = 4$

E	$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})$
$\zeta_E(-3)$	39140592638841172293584598739733993956806941697638400000 0
$h_4(E)$	93937422333218813504603036975361585496336660074332160000000 0

Table 5.6: Computations Using $G \cong (\mathbb{Z}/2\mathbb{Z})^4$ for the Case $n = 4$

E	$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{11})$
$\zeta_E(-3)$	490837513870755658377078988944155986484490506419998073949257728/5
$h_4(E)$	2356020066579627160209979146931948735125554430815990754956437094 40

Chapter 6

The Wild Kernel

As another application of Brauer-Kuroda relations, we consider the motivic wild kernel, which we introduce in this chapter. We begin in Section 6.1 by first describing the wild kernel of a number field E and subsequently considering its analogues in étale and motivic cohomology, defining both the étale wild kernel and the motivic wild kernel, respectively. In Sections 6.2 and 6.3 we use the Poitou-Tate duality sequence to determine an equation for the order of the motivic wild kernel of E, which we use in the Brauer-Kuroda relations in the subsequent chapter.

The background for the material presented in this chapter is primarily taken from [24] and [26]

6.1 Definition of the Wild Kernel

For any number field E, the classical wild kernel $WK_2(E)$ fits in Moore's exact sequence of power norm symbols

$$0 \to WK_2(E) \to K_2(E) \to \oplus_v \mu(E_v) \to \mu(E) \to 0,$$

where v runs through all finite and infinite primes of E, and $\mu(E)$ and $\mu(E_v)$ denote the group of roots of unity in E and E_v respectively. For a fixed prime p, the p-primary

part of $WK_2(E)$, denoted $WK_2(E)(p)$, is the kernel of the map

$$K_2(o_E)(p) \to \oplus_{\upsilon}\mu(E_{\upsilon})(p),$$

where υ runs through all primes in E above p and if $p = 2$, also through the real infinite primes.

In the context of étale cohomology, the analogues of the p-part of the classical wild kernel are the *higher étale wild kernels*, $WK^{(p)}_{2n-2}(E)$, defined as

$$WK^{(p)}_{2n-2}(E) = ker(H^2_{\acute{e}t}(o'_E, \mathbb{Z}_p(n)) \to \oplus_{\upsilon|p} H^2(E_{\upsilon}, \mathbb{Z}_p(n))),$$

where again υ runs through all primes in E above the prime p, and if $p = 2$, also through all real infinite primes. As we shall see, part of the Poitou-Tate duality sequence then yields the exact sequence

$$0 \to WK^{(p)}_{2n-2}(E) \to H^2_{\acute{e}t}(o'_E, \mathbb{Z}_p(n))$$

$$\to \oplus_{\upsilon|p} H^2(E_{\upsilon}, \mathbb{Z}_p(n)) \to H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^{\vee} \to 0,$$

analogous to Moore's exact sequence, where $^{\vee}$ denotes the Pontryagin dual.

Viewing the p-adic étale cohomology groups as the p-part of motivic cohomology, we define the n-th *motivic wild kernel*, $WK^{\mathcal{M}}_{2n-2}(E)$, by

$$WK^{\mathcal{M}}_{2n-2}(E) = \prod_p WK^{(p)}_{2n-2}(E).$$

Similarly, $WK^{\mathcal{M}}_{2n-2}(E)$ fits into its own motivic analogue of Moore's exact sequence, namely

$$0 \to WK^{\mathcal{M}}_{2n-2}(E) \to H^2_{\mathcal{M}}(E, \mathbb{Z}(n))$$

$$\to \oplus_{\upsilon} H^0(E_{\upsilon}, \mathbb{Q}/\mathbb{Z}(1-n))^{\vee} \to H^0(E, \mathbb{Q}/\mathbb{Z}(1-n))^{\vee} \to 0,$$

where the sum extends over all finite and real infinite primes of E if n is even, and over all finite primes if n is odd.

6.2 Poitou-Tate Duality Sequence

Let S be a nonempty set of primes of a number field E containing the infinite primes. Let A be a finite G_S-module and let

$$A' = Hom(A, o_E^{S\times})$$

denote the dual G_S-module of A, where $o_E^{S\times}$ is the group of S-units of o_E^S.

There exists a 9-term long exact sequence called the *Poitou-Tate Duality Sequence*, (cf. [26]) given by

$$\begin{aligned} 0 \to H^0(G_S, A) \to \oplus_{v\in S} H^0(E_v, A) \to H^2(G_S, A')^\vee \to \\ \to H^1(G_S, A) \to \oplus_{v\in S} H^1(E_v, A) \to H^1(G_S, A')^\vee \to \\ \to H^2(G_S, A) \to \oplus_{v\in S} H^2(E_v, A) \to H^0(G_S, A')^\vee \to 0, \end{aligned} \tag{6.1}$$

where $H^i(G_S, A')^\vee = Hom(H^i(G_S, A'), \mathbb{Q}/\mathbb{Z})$ denotes the Pontryagin dual of $H^i(G_S, A')$.

Suppose now S is a finite set of primes above a fixed prime p, including the infinite primes if $p = 2$, and let $A = \mu_{p^m}^{\otimes n}$. Then, the dual G_S-module of A is given by $(\mu_{p^m}^{\otimes n})' = \mu_{p^m}^{\otimes 1-n}$ (cf. [27]), and, as such, the last 3 terms of the Poitou-Tate Duality Sequence now read

$$H^2_{ét}(o'_E, \mu_{p^m}^{\otimes n}) \to \oplus_{v\in S} H^2_{ét}(E_v, \mu_{p^m}^{\otimes n}) \to H^0(E, \mu_{p^m}^{\otimes 1-n})^\vee \to 0. \tag{6.2}$$

As these are all finite abelian groups, passing to the inverse limit over m yields the exact sequence

$$H^2_{ét}(o'_E, \mathbb{Z}_p(n)) \to \oplus_{v\in S} H^2_{ét}(E_v, \mathbb{Z}_p(n)) \to H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee \to 0, \tag{6.3}$$

where the last term follows by

$$\begin{aligned} \varprojlim H^0(E, \mu_{p^m}^{\otimes 1-n})^\vee &\cong \varprojlim Hom(H^0(E, \mu_{p^m}^{\otimes 1-n}), \mathbb{Q}/\mathbb{Z}) \\ &\cong Hom(\varinjlim H^0(E, \mu_{p^m}^{\otimes 1-n}), \mathbb{Q}/\mathbb{Z}) \\ &\cong Hom(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(1-n)), \mathbb{Q}/\mathbb{Z}) \\ &\cong H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee. \end{aligned}$$

We note that the sequence 6.3 remains exact if we reduce the set S to include only the finite primes $\upsilon|p$. By the definition of the motivic wild kernel, the p-part of $WK^{\mathcal{M}}_{2n-2}(E)$ then fits into the exact sequence as follows:

$$0 \to WK^{\mathcal{M}}_{2n-2}(E)(p) \to H^2_{\acute{e}t}(o'_E, \mathbb{Z}_p(n))$$

$$\to \oplus_{\upsilon|p} H^2_{\acute{e}t}(E_\upsilon, \mathbb{Z}_p(n)) \to H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee \to 0. \tag{6.4}$$

6.3 Deriving $h^{\mathcal{M}}_n(E)$

For a finite prime $\upsilon \in S$, local duality yields an isomorphism

$$H^i(E_\upsilon, A) \cong H^{2-i}(E_\upsilon, A')^\vee$$

for $0 \leq i \leq 2$. In our case, this gives

$$H^2_{\acute{e}t}(E_\upsilon, \mathbb{Z}_p(n)) \cong H^0(E_\upsilon, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee,$$

where we have passed to the inverse limit over m.

The groups $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(1-n))$ and $H^0(E_\upsilon, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee$ are finite cyclic for $n \neq 1$. More specifically, since for any t and any m the Galois action on $\mu_{p^m}^{\otimes -t}$ is inverse to the action on $\mu_{p^m}^{\otimes t}$, we have

$$\begin{aligned} |H^0(E_\upsilon, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee| &= |H^0(E_\upsilon, \mathbb{Q}_p/\mathbb{Z}_p(1-n))| \\ &= |H^0(E_\upsilon, \mathbb{Q}_p/\mathbb{Z}_p(n-1))| \\ &= w_{n-1}(E)(p), \end{aligned}$$

and

$$\begin{aligned} |\oplus_{\upsilon|p} H^0_{\acute{e}t}(E_\upsilon, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee| &= |\oplus_{\upsilon|p} H^0_{\acute{e}t}(E_\upsilon, \mathbb{Q}_p/\mathbb{Z}_p(1-n))| \\ &= |\oplus_{\upsilon|p} H^0_{\acute{e}t}(E_\upsilon, \mathbb{Q}_p/\mathbb{Z}_p(n-1))| \\ &= \prod_{\upsilon|p} w_{n-1}(E_\upsilon)(p). \end{aligned}$$

Taking orders, it follows that

$$|WK^{\mathcal{M}}_{2n-2}(E)(p)| = w_{n-1}(E)(p) \cdot \frac{|H^2_{\acute{e}t}(o'_E, \mathbb{Z}_p(n))|}{\prod_{\upsilon|p} w_{n-1}(E_\upsilon)(p)}.$$

Let $h^{\mathcal{M}}_n(E)$ denote the order of the motivic wild kernel of E, and recall that $h_n(E) = |H^2(o_E, \mathbb{Z}(n))| = \prod_p |H^2_{\acute{e}t}(o'_E, \mathbb{Z}_p(n))|$ where p runs through all primes. Keeping in mind that in Section 6.2 we have omitted the prime $p = 2$, we take the product over all primes p to obtain

$$h^{\mathcal{M}}_n(E) = w_{n-1}(E) \cdot \frac{h_n(E)}{|H^2_{\acute{e}t}(o'_E, \mathbb{Z}_2(n))| \cdot \prod_p \prod_{\upsilon|p} w_{n-1}(E_\upsilon)(p)}.$$

Finally, it is shown in [12], that $|H^2_{\acute{e}t}(o'_E, \mathbb{Z}_2(n))| = 2^d$ where d is the degree of the extension E over $\mathbb{Q}$, so that the order of the motivic wild kernel is

$$h^{\mathcal{M}}_n(E) = w_{n-1}(E) \cdot \frac{h_n(E)}{2^d \cdot \prod_p \prod_{\upsilon|p} w_{n-1}(E_\upsilon)(p)}. \tag{6.5}$$

Chapter 7

Brauer Relations for the Motivic Wild Kernel

In Chapter 4, we have seen that the values $w_n(E)$ and the higher class numbers of a totally real number field E vanish in Brauer-Kuroda relations for all n even. In the spirit of this work, we now consider the n-th movitic wild kernel, $WK_{2n-2}^{\mathcal{M}}(E)$, whose order, given by 6.5, depends in part on $h_n(E) \cdot w_{n-1}(E)$.

In this chapter, we verify whether $h_n^{\mathcal{M}}(E)$ also cancels in Brauer-Kuroda relations, in the case that E is totally real and $n \geq 2$ is even. As we shall see, this is largely dependent on the local behaviour of the number field E, and therefor conditional upon the choice of E. Hence, for simplicity, we shall focus only on totally real field extensions $E/\mathbb{Q}$ with multicyclic, dihedral or alternating Galois group.

7.1 Determining Local Factors

Let E be a totally real number field of degree d with Galois group $G = Gal(E/\mathbb{Q})$. Let D denote the discriminant of E, and let $n \geq 2$ be an even integer. Suppose

$$\sum_H a(H) Ind_H^G(\mathbf{1}_H) = 0$$

denotes a Brauer-Kuroda relation on G. Then, by Proposition 4.3 and Theorem 3.1, the order of the motivic wild kernel $h_n^{\mathcal{M}}$ cancels in such a relation if the factors $2^d \cdot \prod_p \prod_{v|p} w_{n-1}(E_v^H)(p)$ also cancel.

It has been shown by [4] that the values 2^d vanish in a Brauer-Kuroda relation, so that whether this also holds for $h_n^{\mathcal{M}}(E)$ is now entirely dependent on $\prod_p \prod_{v|p} w_{n-1}(E_v^H)(p)$. As such, we examine the local behaviour of our fields E^H, noting that the values $w_{n-1}(E_v^H)(p)$ contain only the p-primary part of $w_{n-1}(E_v^H)$, and thus are of the form p^k for some integer $k \geq 0$.

From [12], we have the following properties:

- $\prod_{v|2} w_{n-1}(E_v^H)(2) \leq 2^d$
- For all odd primes $p \nmid D$, $\prod_{v|p} w_{n-1}(E_v^H)(p) = 1$
- For all primes $p \geq 3$, $\prod_{v|p} w_1(E_v^H)(p) \leq p^{\frac{d}{p-1}}$
- If $w_1(E_v^H)(p) = p^k$ for $k \geq 1$, then $\mathbb{Q}_p(\zeta_{p^k}) \subseteq E_v^H$ and hence $e(v|p) \geq p^k - p^{(k-1)}$,

Additionally, we note that the Brauer-Kuroda relations for both the dihedral case and the multicyclic case involve quadratic extensions $F = \mathbb{Q}(\sqrt{m})$, where $m \geq 2$. The local factors $\prod_{v|p} w_1(E_v^H)(p)$ in $h_n^{\mathcal{M}}$ are determined, in part, by the splitting of primes in these quadratic extensions. By [20], a prime p splits in F as follows:

If $p|m$ $\quad po_F = (p, \sqrt{m})^2$

If m is odd
$$2o_F = \begin{cases} (2, 1+\sqrt{m})^2 & \text{if } m \equiv 3 \bmod 4 \\ (2, \frac{1+\sqrt{m}}{2})(2, \frac{1-\sqrt{m}}{2}) & \text{if } m \equiv 1 \bmod 8 \\ \text{prime} & \text{if } m \equiv 5 \bmod 8 \end{cases}$$

If p is odd and $p \nmid m$
$$po_F = \begin{cases} (p, n+\sqrt{m})(p, n-\sqrt{m}) & \text{if } m \equiv n^2 \bmod p \\ \text{prime} & \text{if } m \text{ is not a square mod } 8 \end{cases}$$

Observe also that, in some cases, our local fields will be isomorphic to quadratic extensions of $\mathbb{Q}_p$, for some prime p. In the case $p = 2$, there are exactly 7 distinct quadratic extensions of $\mathbb{Q}_2$, and these may be represented by (cf. [19]):

$$\mathbb{Q}_2(\sqrt{-1}),\ \mathbb{Q}_2(\sqrt{-3}),\ \mathbb{Q}_2(\sqrt{3}),\ \mathbb{Q}_2(\sqrt{2}),\ \mathbb{Q}_2(\sqrt{-2}),\ \mathbb{Q}_2(\sqrt{-6}),\ \mathbb{Q}_2(\sqrt{6}).$$

In this case, beginning with a quadratic field $\mathbb{Q}(\sqrt{m})$, we have that $\mathbb{Q}_2(\sqrt{m}) \cong \mathbb{Q}_2(\sqrt{-1})$ precisely when $m \equiv -1 \mod 8$, in which case $w_1(\mathbb{Q}(\sqrt{m})) = 4$.

7.2 Results

Given a particular Galois extension $Gal(E/\mathbb{Q})$, we present in the following tables the cases in which $\prod_H \prod_{\upsilon|p} w_1(E_\upsilon^H)(p) \neq 1$ for a prime p.

In Table 7.1, the fields $F_{i,\upsilon}$ denote the quadratic subextensions of $\mathbb{Q}$, localized at the primes $\upsilon|2$.

Similarly, in Table 7.2, the fields $K_{i,\upsilon}$ denote the quadratic subextensions of $\mathbb{Q}$, localized at the primes $\upsilon|p$. Here, we note in particular that in the first two cases, all local fields contain at most the 2nd roots of unity.

In Table 7.3, we let $F_{i,\upsilon}$ denote any one of the three subfields of E fixed by the cyclic subgroups of $Gal(E\mathbb{Q}) \cong A_4$ of order 2, localized at the prime $\upsilon|p$. As well, $L_{i,\upsilon}$ denotes the fixed field of any one of the four cyclic subgroups of order 3, localized at the prime υ above p. We note here that this Table is not a complete list, but is presented here to demonstrate that there is in fact, no nice formula expressing the extra factors of $\prod_H \prod_{\upsilon|p} w_1(E_\upsilon^H)(p)$ that do not cancel in the Brauer-Kuroda relation. We note that in Cases $3, 6, 12$ and 13, all local fields contain at most the 2nd roots of unity. We also point out that in Cases $3 - 11$, the choice of fields F_i or L_i can determine whether their corresponding local fields $F_{i,\upsilon}, L_{i,\upsilon}$ contain any roots of unity.

Notably, we observe that whether the local factors $\prod_H \prod_{\upsilon|p} w_1(E_\upsilon^H)(p)$ cancel in a Brauer-Kuroda relation is now heavily influenced by the extension $E/\mathbb{Q}$. Given the many cases presented here, especially in the case of the alternating group, it is clear that the local factors do not behave nicely in the Brauer-Kuroda relations, and as a result, the motivic wild kernels do not, generally, cancel in these relations.

Table 7.1: Computations On Local Factors for Multicyclic Galois Extensions for $n = 2$

$Gal(E/\mathbb{Q})$	p	$\prod_H \prod_{v\mid p} w_1(E_v^H)(p)$	Case	Local Factors
$(\mathbb{Z}/2\mathbb{Z})^2$	2	2	$2o_E = \mathfrak{P}^4$	$E_v \supseteq \mathbb{Q}_2(\zeta_8)$ $F_{1,v} \cong \mathbb{Q}_2(i)$
$(\mathbb{Z}/2\mathbb{Z})^3$	2	2	$2o_E = \mathfrak{P}^8$	$E_v \supseteq \mathbb{Q}_2(\zeta_8)$ $F_{1,v} \cong \mathbb{Q}_2(i)$
	2	2	$2o_E = \mathfrak{P}^4$	$E_v \supseteq \mathbb{Q}_2(\zeta_8)$ $F_{1,v} \cong \mathbb{Q}_2(i)$ $F_{2,v} \cong \mathbb{Q}_2(\sqrt{2})$
	2	4	$2o_E = \mathfrak{P}_1^4 \cdot \mathfrak{P}_2^4$	$E_{v_1} \supseteq \mathbb{Q}_2(\zeta_8)$ $F_{1,v} \cong \mathbb{Q}_2(i)$ $F_{2,v} \cong \mathbb{Q}_2(i)$ $F_{3,v} \cong \mathbb{Q}_2(\sqrt{2})$

Table 7.2: Computations On Local Factors for $Gal(E/\mathbb{Q}) \cong D_{2\ell}$ for $n = 2$

p	$\prod_H \prod_{v\mid p} w_1(E_v^H)(p)$	Case	Local Factors
2	$2^{\ell-1}$	$2o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2 \cdots \mathfrak{P}_\ell^2$	-
2	$2^{\ell-1}$	$2o_E = \mathfrak{P}_1\mathfrak{P}_2 \cdots \mathfrak{P}_\ell$	-
3	$3^{\ell-1}$	$3o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2 \cdots \mathfrak{P}_\ell^2$	$E_v \supseteq \mathbb{Q}_3(\zeta_3)$ $K_v \cong \mathbb{Q}_3(\zeta_3)$

Table 7.3: Computations On Local Factors for $Gal(E/\mathbb{Q}) \cong A_4$ for $n = 2$

Case	p	$\prod_H \prod_{v\mid p} w_1(E_v^H)(p)$	Case	Local Factors
1	2	2	$2o_E = \mathfrak{P}^{12}$	$E_v \supseteq \mathbb{Q}_2(i)$
2	3	3	$3o_E = \mathfrak{P}^{12}$	$E_v \supseteq \mathbb{Q}_3(\zeta_3)$
3	2	2^3	$2o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2\cdots\mathfrak{P}_6^2$	-
4	2	2^2	$2o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2\cdots\mathfrak{P}_6^2$	$F_{1,v} \supseteq \mathbb{Q}_2(i)$
5	2	2	$2o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2\cdots\mathfrak{P}_6^2$	$F_{1,v} \supseteq \mathbb{Q}_2(i)$ $F_{2,v} \supseteq \mathbb{Q}_2(i)$
6	2	2^2	$2o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2\cdots\mathfrak{P}_6^2$	-
7	2	2	$2o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2\cdots\mathfrak{P}_6^2$	$L_{1,v} \supseteq \mathbb{Q}_2(i)$
8	3	3^3	$3o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2\cdots\mathfrak{P}_6^2$	$F_{1,v} \supseteq \mathbb{Q}_3(\zeta_3)$
9	3	3^2	$3o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2\cdots\mathfrak{P}_6^2$	$F_{1,v} \supseteq \mathbb{Q}_3(\zeta_3)$ $F_{2,v} \supseteq \mathbb{Q}_3(\zeta_3)$
10	3	3	$3o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2\cdots\mathfrak{P}_6^2$	$L_{1,v} \supseteq \mathbb{Q}_3(\zeta_3)$
11	3	3^2	$3o_E = \mathfrak{P}_1^2\mathfrak{P}_2^2\cdots\mathfrak{P}_6^2$	$L_{1,v} \supseteq \mathbb{Q}_3(\zeta_3)$ $L_{2,v} \supseteq \mathbb{Q}_3(\zeta_3)$
12	2	2	$2o_E = \mathfrak{P}_1\mathfrak{P}_2\cdots\mathfrak{P}_6$	-
13	2	2^2	$2o_E = \mathfrak{P}_1\mathfrak{P}_2\cdots\mathfrak{P}_6$	-

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